Physics 523, Quantum Field Theory II

Homework 9

Due Wednesday, 17$^{\text{th}}$ March 2004

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β-Functions in Pseudo-Scalar Yukawa Theory

Let us consider the massless pseudo-scalar Yukawa theory governed by the renormalized Lagrangian,

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \overline{\psi}i\not\partial\psi - ig\overline{\psi}\gamma^5\psi\phi - \frac{\lambda}{4!}\phi^4$$
$$+\frac{1}{2}\delta_\phi(\partial_\mu\phi)^2\overline{\psi}i\delta_\psi\not\partial\psi - ig\delta_g\overline{\psi}\gamma^5\psi\phi - \frac{\delta_\lambda}{4!}\phi^4.$$

In homework 8, we calculated the divergent parts of the renormalization counterterms $\delta_\phi, \delta_\psi, \delta_g$, and δ_λ to 1-loop order. These were shown to be

$$\delta_\phi = -\frac{g^2}{8\pi^2}\log\frac{\Lambda^2}{M^2}, \qquad \delta_\psi = -\frac{g^2}{32\pi^2}\log\frac{\Lambda^2}{M^2};$$
$$\delta_\lambda = \left(\frac{3\lambda^2}{32\pi^2} - \frac{3g^4}{2\pi^2}\right)\log\frac{\Lambda^2}{M^2}, \qquad \delta_g = \frac{g^2}{16\pi^2}\log\frac{\Lambda^2}{M^2}.$$

Using the definitions of B_i and A_i in Peskin and Schroeder, these imply that

$$A_\phi = -\gamma_\phi = -\frac{g^2}{8\pi^2}, \qquad A_\psi = -\gamma_\psi = -\frac{g^2}{32\pi^2};$$
$$B_\lambda = \frac{3g^4}{2\pi^2} - \frac{3\lambda^2}{32\pi^2} \qquad B_g = -\frac{g^2}{16\pi^2}.$$

Therefore, we see that

$$\beta_g = -2gB_g - 2gA_\psi - gA_\phi = 2g\frac{g^2}{16\pi^2} + 2g\frac{g^2}{32\pi^2} + g\frac{g^2}{8\pi^2} = \frac{5g^3}{16\pi^2};$$
$$\beta_\lambda = -2B_\lambda - 4\lambda A_\phi = 2\left(\frac{3\lambda^2}{32\pi^2} - \frac{3g^4}{2\pi^2}\right) + 4\lambda\frac{g^2}{8\pi^2} = \frac{3\lambda^2 + 8\lambda g^2 - 48g^4}{16\pi^2}$$

While it was supposedly unnecessary, the running couplings were computed to be[1],

$$\overline{g}(p) = \sqrt{\frac{16\pi^2}{1 - 10\log p/M}};$$
$$\overline{\lambda}(p) = \overline{\lambda} = \frac{\overline{g}^2}{3}\left(1 + \sqrt{145}\frac{-\frac{4\sqrt{145}+149}{141} + \overline{g}^{2\sqrt{145}/5}}{-\frac{4\sqrt{145}+149}{141} - \overline{g}^{2\sqrt{145}/5}}\right).$$

Notice that both $\overline{g}$ and $\overline{\lambda}$ generally become weaker at large distances because for typical values of g, λ we see that β_g and β_λ are both positive. However, if $\lambda << g$ then β_λ will be negative and so $\overline{\lambda}$ will grow stronger at larger distances. Near small values of g and λ the theory shows interesting interplay between g and λ. Also interesting is the characteristic Landau pole in $\overline{\lambda}$ suggesting that we should not trust this theory at too large a scale.

Below is a graph of $\overline{g}$ versus $-\overline{\lambda}$ indicating the direction of Renormalization Group flow as the interaction distance grows larger.

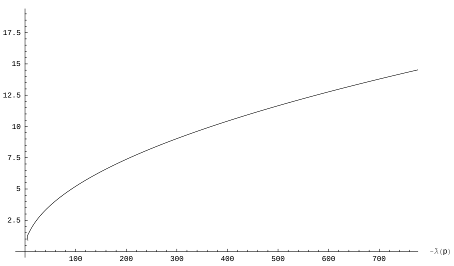

Figure 1. Renormalization Group Flow as a funciton of scale. Arrow indicates flow in the direction of larger distances. For this plot, M was taken to be 10^4.

[1]See appendix.

Minimal Subtraction

Let us define the β-function as it appears in dimensional regularization as

$$\beta(\lambda,\epsilon) = M\frac{d}{dM}\lambda\bigg|_{\lambda_0,\epsilon},$$

where it is understood that $\beta(\lambda) = \lim_{\epsilon\to 0}\beta(\lambda,\epsilon)$. We notice that the bare coupling is given by $\lambda_0 = M^\epsilon Z_\lambda(\lambda,\epsilon)\lambda$ where Z_λ is given by an expansion series in ϵ,

$$Z_\lambda(\lambda,\epsilon) = 1 + \sum_{\nu=1}\frac{a_\nu(\lambda)}{\epsilon^\nu}.$$

We are to demonstrate the following.

a) Let us show that Z_λ satisfies the identity $(\beta(\lambda,\epsilon)+\epsilon\lambda)Z_\lambda + \beta(\lambda\epsilon)\lambda\frac{dZ_\lambda}{d\lambda} = 0$.

proof: Noting the general properties of differentiation from elementary analysis, we will proceed by direct computation.

$$\begin{aligned}(\beta(\lambda,\epsilon)+\epsilon\lambda)\,Z_\lambda + \beta(\lambda,\epsilon)\lambda\frac{dZ_\lambda}{d\lambda} &= \beta(\lambda,\epsilon)Z_\lambda + \epsilon\lambda Z_\lambda + \beta(\lambda,\epsilon)\frac{d(Z_\lambda\lambda)}{d\lambda} - \beta(\lambda,\epsilon)Z_\lambda,\\ &= \epsilon\lambda Z_\lambda + M\frac{d\lambda}{dM}\bigg|_{\lambda_0,\epsilon}\frac{d(\lambda_0 M^{-\epsilon})}{d\lambda},\\ &= \epsilon\lambda Z_\lambda - \epsilon M\lambda_0 M^{-\epsilon-1},\\ &= \epsilon\lambda Z_\lambda - \epsilon M^{1+\epsilon}M^{-\epsilon-1}Z_\lambda\lambda,\\ &= 0.\end{aligned}$$

$$\boxed{\therefore (\beta(\lambda,\epsilon)+\epsilon\lambda)Z_\lambda + \beta(\lambda\epsilon)\lambda\frac{dZ_\lambda}{d\lambda} = 0.}$$

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b) Let us show that $\beta(\lambda,\epsilon) = -\epsilon\lambda + \beta(\lambda)$.

proof: We have demonstrated in part (a) above that $(\beta(\lambda,\epsilon)+\epsilon\lambda)Z_\lambda + \beta(\lambda\epsilon)\lambda\frac{dZ_\lambda}{d\lambda} = 0$. Dividing this equation by Z_λ and rearranging terms and expanding in Z_λ, we obtain

$$\begin{aligned}\beta(\lambda,\epsilon)+\epsilon\lambda &= -\beta(\lambda,\epsilon)\frac{\lambda}{Z_\lambda}\frac{dZ_\lambda}{d\lambda},\\ &= -\beta(\lambda,\epsilon)\frac{\lambda}{Z_\lambda}\left(\frac{1}{\epsilon}\frac{da_1}{d\lambda} + \frac{1}{\epsilon^2}\frac{da_2}{d\lambda} + \cdots\right),\\ &= -\beta(\lambda,\epsilon)\lambda\left(\frac{1}{\epsilon}\frac{da_1}{d\lambda} + \frac{1}{\epsilon^2}\frac{da_2}{d\lambda} + \cdots\right)\left(1 - \frac{a_1}{\epsilon} + \cdots\right).\end{aligned}$$

Now, we know that $\beta(\lambda,\epsilon)$ must be regular in ϵ as $\epsilon\to 0$ and so we may expand it as a (terminating)[2] power series $\beta(\lambda,\epsilon) = \beta_0 + \beta_1\epsilon + \beta_2\epsilon^2 + \cdots + \beta_n\epsilon^n$. We notice that $\beta(\lambda) = \beta_0$ in this notation. Let us consider the limit of $\epsilon\to\infty$.

For any $n>0$, we see that the order of the polynomial on the left hand side has degree n whereas the polynomial on the left hand side has degree $n-1$ because as $\epsilon\to\infty$, the equation becomes $\beta_n\epsilon^n = -\beta_n\epsilon^n\lambda\frac{1}{\epsilon}\frac{da_1}{d\lambda}$. But this is a contradiction. —×—

Therefore, both the right and left hand sides must have degree less than or equal to 0. Furthermore, because the left hand side is $\beta(\lambda,\epsilon)+\epsilon\lambda = \beta_0 + \beta_1\epsilon + \epsilon\lambda$ must have degree zero, we see that $\beta_1 = -\epsilon$.

So, expanding $\beta(\lambda,\epsilon)$ as a power series of ϵ, we obtain,

$$\boxed{\therefore \beta(\lambda,\epsilon) = -\epsilon\lambda + \beta(\lambda).}$$

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[2]Professor Larsen does note believe this to be necessary. However, we have been unable to demonstrate the required identity without assuming a terminating power series.

c.i) Let us show that $\beta(\lambda) = \lambda^2 \frac{da_1}{d\lambda}$.

proof: By rewriting the identity obtained from part (a) above and expanding in Z_λ we see that

$$(\beta(\lambda,\epsilon) + \epsilon\lambda)\, Z_\lambda = -\beta(\lambda,\epsilon)\lambda \frac{dZ_\lambda}{d\lambda},$$
$$(\beta(\lambda,\epsilon) + \epsilon\lambda)\left(1 + \frac{a_1}{\epsilon} + \cdots\right) = -\beta(\lambda,\epsilon)\lambda\left(\frac{1}{\epsilon}\frac{da_1}{d\lambda} + \cdots\right).$$

We see that because there is no term on the right hand side of order ϵ^0, it must be that $\beta(\lambda,\epsilon) + \lambda a_1 = 0$ which implies that $\beta(\lambda,\epsilon) = -\lambda a_1$. Furthermore, by equating the coefficients of $\frac{1}{\epsilon^n}$, we have in general that $\beta(\lambda,\epsilon)a_n + \lambda a_{n+1} = -\beta(\lambda,\epsilon)\lambda\frac{da_n}{d\lambda}$. By rearranging terms and using noticing the chain rule of differentiation, we see that this implies that

$$\lambda a_{n+1} = -\beta(\lambda,\epsilon)\left(\lambda\frac{da_n}{d\lambda} + a_n\right) = -\beta(\lambda,\epsilon)\frac{d(\lambda a_n)}{d\lambda}.$$

This fact will be important to the proof immediately below.

Now, by the result of part (b) above, we know that

$$\beta(\lambda)Z_\lambda = (\beta(\lambda,\epsilon) + \epsilon\lambda)Z_\lambda = -\beta(\lambda,\epsilon)\lambda\frac{dZ_\lambda}{d\lambda},$$
$$\beta(\lambda)\left(1 + \frac{a_1}{\epsilon} + \cdots\right) = (\beta(\lambda,\epsilon) + \epsilon\lambda)Z_\lambda = -\beta(\lambda,\epsilon)\lambda\left(\frac{1}{\epsilon}\frac{da_1}{d\lambda} + \cdots\right).$$

Equating the coefficients of terms of order $\frac{1}{\epsilon}$ on the far left and right sides, we see that

$$\beta(\lambda)a_1 = -\beta(\lambda,\epsilon)\lambda\frac{da_1}{d\lambda}.$$

Now, using our result from before that $\beta(\lambda,\epsilon) = -\lambda a_1$, we see directly that

$$\boxed{\therefore \beta(\lambda) = \lambda^2\frac{da_1}{d\lambda}.}$$

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c.ii) Let us show that $\beta(\lambda)\frac{d(\lambda a_\nu)}{d\lambda} = \lambda^2\frac{da_{\nu+1}}{d\lambda}$.

proof: By our result in part (b) above, we have that

$$\beta(\lambda) = (\beta(\lambda,\epsilon) + \epsilon\lambda),$$
$$\therefore \beta(\lambda)\frac{d(Z_\lambda\lambda)}{d\lambda} = (\beta(\lambda,\epsilon) + \epsilon\lambda)\frac{d(Z_\lambda\lambda)}{d\lambda},$$
$$\beta(\lambda)\left(1 + \frac{1}{\epsilon}\frac{d(\lambda a_1)}{d\lambda} + \cdots\right) = (\beta(\lambda,\epsilon) + \epsilon\lambda)\left(1 + \frac{1}{\epsilon}\frac{d(\lambda a_1)}{d\lambda} + \cdots\right).$$

Equating the coefficients of $\frac{1}{\epsilon^\nu}$ on both sides, we see that by using the identities shown above,

$$\begin{aligned}\beta(\lambda)\frac{d(\lambda a_\nu)}{d\lambda} &= \beta(\lambda,\epsilon)\frac{d(\lambda a_\nu)}{d\lambda} + \lambda\frac{d(\lambda a_{\nu+1})}{d\lambda},\\ &= \beta(\lambda,\epsilon)\frac{d(\lambda a_\nu)}{d\lambda} + \lambda^2\frac{da_{\nu+1}}{d\lambda} + \lambda a_{\nu+1},\\ &= \beta(\lambda,\epsilon)\frac{d(\lambda a_\nu)}{d\lambda} + \lambda^2\frac{da_{\nu+1}}{d\lambda} - \beta(\lambda,\epsilon)\frac{d(\lambda a_\nu)}{d\lambda},\\ &= \lambda^2\frac{da_{\nu+1}}{d\lambda}.\end{aligned}$$

So we see in general that

$$\boxed{\therefore \beta(\lambda)\frac{d(\lambda a_\nu)}{d\lambda} = \lambda^2\frac{da_{\nu+1}}{d\lambda}.}$$

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In the minimal subtraction scheme, we define the mass renormalization by $m_0^2 = m^2 Z_m$ where

$$Z_m = 1 + \sum_{\nu=1} \frac{b_\nu}{\epsilon^\nu}.$$

Similarly, we will define the associated β-function $\beta_m(\lambda) = m\gamma_m(\lambda)$ which is given by

$$\beta_m(\lambda) = M \frac{dm}{dM}\bigg|_{m_0,\epsilon}.$$

d.i) Let us show that $\gamma_m(\lambda) = \frac{\lambda}{2}\frac{db_1}{d\lambda}$.

proof: Because m_0^2 is a constant, we know that $\frac{dm_0^2}{dM} = 0$. Therefore, writing $m_0^2 = m^2 Z_m$ we see that this implies

$$\begin{aligned}
\frac{dm_0^2}{dM} = 0 &= 2Z_m m \frac{dm}{dM} + m^2 \frac{dZ_m}{dM}, \\
&= 2Z_m m \frac{\beta_m(\lambda)}{M} + m^2 \frac{dZ_m}{d\lambda}\frac{d\lambda}{dM} = 0; \\
\therefore 0 &= 2Z_m\beta_m(\lambda) + mM\frac{d\lambda}{dM}\frac{dZ_m}{d\lambda}; \\
\therefore 2\beta_m(\lambda)Z_m &= -m\beta(\lambda,\epsilon)\frac{dZ_m}{d\lambda}, \\
2\beta_m(\lambda)\left(1 + \frac{b_1}{\epsilon} + \cdots\right) &= -m\beta(\lambda,\epsilon)\left(\frac{1}{\epsilon}\frac{db_1}{d\lambda} + \cdots\right), \\
2\beta_m(\lambda)\left(1 + \frac{b_1}{\epsilon} + \cdots\right) &= -m\left(\beta(\lambda) - \epsilon\lambda\right)\left(\frac{1}{\epsilon}\frac{db_1}{d\lambda} + \cdots\right),
\end{aligned}$$

We see that the coefficient of the ϵ^0 term on the left hand side is $2\beta_m(\lambda)$ and on the right hand side it is $m\lambda\frac{db_1}{d\lambda}$. Therefore, because these terms must be equal, we see that

$$\beta_m(\lambda) = m\frac{\lambda}{2}\frac{db_1}{d\lambda},$$

$$\boxed{\therefore \gamma_m(\lambda) = \frac{\lambda}{2}\frac{db_1}{d\lambda}.}$$

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d.ii) Let us prove that $\lambda\frac{db_{\nu+1}}{d\lambda} = 2\gamma_m(\lambda)b_\nu + \beta(\lambda)\frac{db_\nu}{d\lambda}$.

proof: Continuing our work from part (d.i) above, we have that

$$2\beta_m(\lambda)\left(1 + \frac{b_1}{\epsilon} + \cdots\right) = -m\left(\beta(\lambda) - \epsilon\lambda\right)\left(\frac{1}{\epsilon}\frac{db_1}{d\lambda} + \cdots\right).$$

It must be that the coefficients of $\frac{1}{\epsilon^\nu}$ are equal on both sides. Therefore, we see that

$$\begin{aligned}
2\beta_m(\lambda)b_\nu &= -m\beta(\lambda)\frac{db_\nu}{d\lambda} + m\lambda\frac{db_{\nu+1}}{d\lambda}, \\
2m\gamma_m(\lambda)b_\nu &= -m\beta(\lambda)\frac{db_\nu}{d\lambda} + m\lambda\frac{db_{\nu+1}}{d\lambda}, \\
\therefore 2\gamma_m(\lambda)b_\nu &= -\beta(\lambda)\frac{db_\nu}{d\lambda} + \lambda\frac{db_{\nu+1}}{d\lambda}.
\end{aligned}$$

Rearranging terms, we see that

$$\boxed{\therefore \lambda\frac{db_{\nu+1}}{d\lambda} = 2\gamma_m(\lambda)b_\nu + \beta(\lambda)\frac{db_\nu}{d\lambda}.}$$

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Appendix

Calculation of the Running Couplings $\overline{g}$ and $\overline{\lambda}$

Let us now solve for the flow of the coupling constants g, λ. We have in general that solutions to the Callan-Symanzik equation will satisfy

$$\frac{d\overline{g}}{d\log p/M} = \beta_g = \frac{5g^3}{16\pi^2} + \mathcal{O}(g^5).$$

This is an ordinary differential equation. We see that

$$-\frac{1}{2}\frac{1}{\overline{g}^2} = \frac{5}{16\pi^2}\log p/M + C,$$

and so

$$\therefore \overline{g}^2(p) = -\frac{8\pi^2}{5\log p/M + C}.$$

The constant C is found so that $g(p = M) = 1$.[3] This yields $C = -1/2$.

To find the flow of λ, however, it will be convenient to introduce a new variable $\eta \equiv \lambda/g^2$. We must then solve the equation

$$\frac{d\overline{\eta}}{d\log p/M} = \frac{\beta_\lambda}{g^2} - 2\frac{\lambda\beta_g}{g^3} = \frac{\left(3\eta^2 - 2\eta - 48\right)g^2}{16\pi^2} + \mathcal{O}(g^4).$$

This is again a simple ordinary differential equation. We see that this implies

$$\int\frac{d\overline{\eta}}{3\eta^2 - 2\eta - 48} = \int\frac{g^2}{16\pi^2}d\log p/M.$$

Note that from our work above, $\frac{g^2}{16\pi^2}d\log p/M = \frac{g^2}{16\pi^2}d\left(-\frac{8\pi^2}{5g^2}\right) = \frac{1}{5g}dg$. Therefore,

$$\int\frac{d\overline{\eta}}{3\eta^2 - 2\eta - 48} = \int\frac{1}{5g}dg.$$

And so,

$$\log\left(\frac{3\overline{\eta} - \sqrt{145} - 1}{3\overline{\eta} + \sqrt{145} - 1}\right) = \frac{2\sqrt{145}}{5}\log g + C.$$

Solving this equation in terms of η, we see that we have

$$\begin{aligned}
\overline{\eta} &= \frac{Cg^{2\sqrt{145}/5}\left(\sqrt{145} - 1\right) + \sqrt{145} + 1}{3 - 3Cg^{2\sqrt{145}/5}}, \\
&= \frac{1 - Cg^{2\sqrt{145}/5}}{3 - 3Cg^{2\sqrt{145}/5}} + \frac{Cg^{2\sqrt{145}/5}\sqrt{145} + \sqrt{145}}{3 - 3Cg^{2\sqrt{145}/5}}, \\
&= \frac{1}{3}\left(1 + \sqrt{145}\frac{C + g^{2\sqrt{145}/5}}{C - g^{2\sqrt{145}/5}}\right).
\end{aligned}$$

$$\therefore \overline{\lambda} = \frac{g^2}{3}\left(1 + \sqrt{145}\frac{C + g^{2\sqrt{145}/5}}{C - g^{2\sqrt{145}/5}}\right).$$

As before, the constant term C is found by requiring that $\overline{\lambda}(p = M) = 1$. The constant is then $C = -\frac{4\sqrt{145}+149}{141}$.

[3]It can be argued that this is a poor choice of C because it requires the reference scale to be non-perturbative. Nevertheless, it is not a free parameter.