A Characterisation of Smoothness for Freud Weights

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24 September, 1998

Abstract

We obtain a new characterisation of smoothness for weighted polynomial approximation with respect to Freud weights together with an existence theorem for derivatives. Our methods rely heavily on realisation functionals.

AMS(MOS) Classification: 41A10, 42C05
Keywords and Phrases: Freud Weight, Jackson-Bernstein Theorem, Modulus of Smoothness, Realization functional, Polynomial Approximation.

1 Introduction

Recently, there has been much interest in the study of rates of polynomial approximation in weighted $L_p(0 < p \leq \infty)$ spaces, associated with fast decaying weights on the real line and $[-1, 1]$. We refer the reader to [1-5], [8-11] and the references cited therein, for a detailed and comprehensive account of the above topic.

In this paper, we obtain a new characterisation of smoothness in $L_p(1 \leq p \leq \infty)$ for weighted polynomials associated with Freud weights on the real line complementing earlier work of [3], [4], [9] and prove an existence theorem for derivatives in $L_p(0 < p \leq \infty)$. In order to state our results, we need to define our class of weight functions and various quantities. First we say that a real valued function $f : (a, b) \longrightarrow (0, \infty)$ is *quasi increasing* if there exists a positive constant C such that

$$a < x < y < b \Longrightarrow f(x) \leq Cf(y).$$

Our weight class will assumed to be admissible in the sense of the following definition.

Definition 1.1

Let

$$W = \exp(-Q)$$

where $Q : \mathbb{R} \longrightarrow \mathbb{R}$ is even and continuous. Then W is an admissible Freud weight and we shall write $W \in \mathcal{E}$ if the following conditions below hold.

(a) *Q' exists and is positive in $(0, \infty)$.*

(b) *$xQ'(x)$ is strictly increasing in $(0, \infty)$ with*

$$\lim_{|x| \to 0^+} xQ'(x) = 0.$$

(c) *For some $\lambda > 1$, $A > 1$, $B > 1$ and $C > 0$,*

$$A \leq \frac{Q'(\lambda x)}{Q'(x)} \leq B,\, x \geq C. \tag{1.1}$$

Remark 1.2

(a) The archetypal example of our class of weights is

$$W_\lambda(x) := \exp(-(|x|^\lambda)), x \in \mathbb{R}. \tag{1.2}$$

Here, in particular, $A = B = \lambda^\lambda$.

(b) (1.1) first appeared in [8]. It implies the more frequently used condition, see [12,13],

$$A_1 \leq \frac{xQ'(x)}{Q(x)} \leq B_1,\, x \geq C_1.$$

for positive constants A_1, B_1 and C_1.

Armed with the above class of admissible Freud weights above, we now define a suitable measure of weighted distance.

Let $I \subseteq \mathbb{R}$ be an interval and

$$L_{p,W}(I) := \{f : I \longrightarrow \mathbb{R} : fW \in L_p(I), 0 < p \leq \infty\}$$

where if $p=\infty$, f is further continuous and satisfies

$$\lim_{|x|\to\infty} fW(x)=0.$$

We equip $L_{p,W}(I)$ with the quasi norm

$$\|fW\|_{L_p(I)} := \begin{cases} \left(\int_I |fW|^p(x)dx\right)^{1/p} & ,0<p<\infty \\ \sup_{x\in I}|fW|(x) & ,p=\infty \end{cases}$$

and interpret $(L_{p,W}(I), \|;\|)$ as a metric space in the usual way. In particular, taking $I=\mathbb{R}$, we may define the $L_p(0<p\le\infty)$ error in best weighted polynomial approximation by:

$$E_n[f]_{W,p} := \inf_{P\in\mathcal{P}_n} \|(f-P)W\|_{L_p(\mathbb{R})}, f\in L_{p,W}(\mathbb{R}) \tag{1.3}$$

where $\mathcal{P}_n$ denotes the class of polynomials of degree at most $n\ge 1$.

In [8], Jackson and Bernstein estimates for $E_n[f]$ for fixed $f\in L_{p,W}(0<p\le\infty)$ were investigated. In order to describe these results, we need the notion of the Mhaskar-Rakhmanov-Saff number and a suitable weighted modulus of smoothness which we define below.

Mhaskar-Rakhmanov-Saff number

Let $W\in\mathcal{E}$ and define the Mhaskar-Rakhmanov-Saff number, $a_u, u\ge 0$ by the equation:

$$u=\frac{2}{\pi}\int_0^1 \frac{a_u t Q'(a_u t)}{\sqrt{1-t^2}}dt, u>0.$$

For those who are not familiar, we quickly recall that its significance lies partly in the identity, see [12,13],

$$\|PW\|_{L_\infty[-a_n,a_n]} = \|PW\|_{L_\infty(\mathbb{R})},\ P\in\mathcal{P}_n,\ n\ge 1.$$

Under our assumptions on Q, it was shown in [8] that a_u is uniquely defined, is a strictly increasing function of u and is continuous for $u\in(0,\infty)$. For example for W_λ, $a_u=Cu^{1/\lambda}$ for some $C>0$ independent of u.

The Weighted Jackson Modulus of Continuity

The following weighted Jackson modulus of continuity for Freud weights was introduced and studied in [8].

Definition 1.3

Let $W \in \mathcal{E}$, $0 < p \le \infty$, $f \in L_{p,W}(\mathbb{R})$, $r \ge 1$ *and set:*

$$\begin{aligned}\omega_{r,p}(f,W,t) := & \sup_{0<h\le t} \|\Delta_h^r(f,x,\mathbb{R})\|_{L_p(|x|\le\sigma(h))} \\ & + \inf_{R\in\mathcal{P}_{r-1}} \|(f-R)W\|_{L_p(|x|\ge\sigma(t))}.\end{aligned} \tag{1.4}$$

Here

$$\sigma(t) := \inf\left\{a_u : \frac{a_u}{u} \le t\right\}, t > 0 \tag{1.5}$$

and for a real interval J,

$$\Delta_h^r(f,x,J) := \begin{cases} \sum_{i=0}^r \binom{r}{i}(-1)^i f(x+\frac{rh}{2}-ih) & , x \pm \frac{rh}{2} \in J \\ 0 & , \text{otherwise} \end{cases}$$

is the r*th symmetric difference of* f.

Remark 1.4

(a) The essential feature of the function σ in (1.5) is that it satisfies the following important condition. Uniformly for $n \ge 1$, there exist constants $C_j > 0$, $j = 1, 2$ such that

$$C_1 \le \frac{\sigma\left(\frac{a_n}{n}\right)}{a_n} \le C_2.$$

Thus, in a sense, $\sigma(\frac{a_n}{n})$ serves as the inverse of the function

$$a_n :\longrightarrow \frac{a_n}{n}, n \ge 1.$$

Typically, t is small and will be taken as $\frac{a_n}{n}$ for $n \ge n_0$ for some fixed but large enough n_0. This latter quantity always tends to zero for large n for our class of admissible weights, see (3.3).

(b) The tail of the modulus $\omega_{r,p}(f,W,;)$ reflects the inability of weighted polynomials (PW), $P \in \mathcal{P}_n$ to approximate well beyond $[-a_n, a_n]$. Its presence ensures that for $f \in \mathcal{P}_{r-1}$, $r \ge 1$,

$$\omega_{r,p}(f,W,;) \equiv 0.$$

(c) Traditionally for Erdős weights on $\mathbb{R}$ and non Szegő weights on $[-1,1]$, see [1,11], the increment h in the main part of the modulus in (1.4) depends on x to allow for endpoint effects in $[-a_n, a_n]$ much as in the classical Ditzian-Totik modulus on $[-1,1]$ which admits a factor of $\sqrt{1-x^2}$. This is not the case for Freud weights on the real line.

We finish this section with two important theorems which were established in [8,9]. In order to state them, we adopt the following convention that will be used in the sequel.

Throughout, for real sequences $\{A_n\}$ and $\{B_n\} \neq 0$

$A_n = O(B_n)$ and $A_n \sim B_n$ will mean respectively that there exist positive constants C_1, C_2 and C_3 independent of n such that $\frac{A_n}{B_n} \leq C_1$ and $C_2 \leq A_n/B_n \leq C_3$.

Similar notation will be used for functions and sequences of functions.

Theorem 1.5

Let $W \in \mathcal{E}$, $0 < p \leq \infty$, $f \in L_{p,W}(\mathbb{R})$, $r \geq 1$ *and* $n \geq n_0$. *Assume that there is a Markov-Bernstein inequality of the form*

$$\|R'W\|_{L_p(\mathbb{R})} \leq C_1 \frac{n}{a_n} \|RW\|_{L_p(\mathbb{R})},\ R \in \mathcal{P}_n. \tag{1.6}$$

Then there exists $C_1 > 0$ *independent of* f *and* n *such that*

$$E_n[f]_{W,p} \leq C_1 w_{r,p}(f, W, \frac{a_n}{n}). \tag{1.7}$$

Moreover, if $p \geq 1$, *we may dispense with the assumption* (1.6).

In order to establish (1.7), the following realisation functional was used which we define below.

Set:

$$K_{r,p}(f, W, t^r) := \inf_{P \in \mathcal{P}_n} \left\{ \|(f-P)W\|_{L_p(\mathbb{R})} + t^r \|P^{(r)}W\|_{L_p(\mathbb{R})} \right\}. \tag{1.8}$$

Here t is chosen in advance and n depends on t by the relation:

$$n = n(t) := \inf\{k : \frac{a_k}{k} \leq t\}. \tag{1.9}$$

The concept of realization should be attributed to Hristov and Ivanov [7]. It enabled the authors in [8] to use a general technique of Ditzian, Hristov and Ivanov [7] to show:

Theorem 1.6

Let $W \in \mathcal{E}$, $0 < p \leq \infty$, $f \in L_{p,W}(\mathbb{R})$, $r \geq 1$, $\alpha > 0$ *and assume* (1.6). *Let* $t \in (0, D)$ *where* D *is a small enough fixed positive number and determine* n *by* (1.9). *Then uniformly for* f *and* t *the following hold:*

(a)

$$\omega_{r,p}(f, W, t) \sim K_{r,p}(f, W, t^r). \tag{1.10}$$

(b)

$$\omega_{r,p}(f,W,t) \sim \omega_{r,p}(f,W,\alpha t) \sim \omega_{r,p}\left(f,W,\frac{a_n}{n}\right). \tag{1.11}$$

(c)

$$\begin{aligned} &K_{r,p}(f,W,t^r) \\ &\sim \|(f-P_n^*)W\|_{L_p(\mathbb{R})} + t^r\|P_n^{*(r)}W\|_{L_p(\mathbb{R})}. \end{aligned} \tag{1.12}$$

Here, $P_{n,p}^*(f)=P_n^*(f)$ *is the best approximant to* f *from* $\mathcal{P}_n$ *satisfying*

$$\|(f-P_n^*)W\|_{L_p(\mathbb{R})} = E_n[f]_{W,p}. \tag{1.13}$$

(d) *If* $1 \le p \le \infty$ *and* f *satisfies the extra smoothness requirement*

$$f^r W \in L_p(\mathbb{R})$$

then there exists $C_1 > 0$ *independent of* t *and* f *such that*

$$\omega_{r,p}(f,W,t) \le C_1 t^r \|f^{(r)}W\|_{L_p(\mathbb{R})}. \tag{1.14}$$

(e) *Moreover if in parts* a − c *above we only assume* $p \ge 1$*, then the results hold without the assumption* (1.6)*.*

This paper is organized as follows: In Section 2, we present our main results and in Section 3, we establish Theorem 2.2, Theorem 2.3, Theorem 2.5 and Theorem 2.6.

2 Statements of Results

Throughout this paper, C, C_1,... will denote positive constants independent of t, n, x and $P \in \mathcal{P}_n$ while the symbol D will always denote a small enough but fixed positive constant. The same symbol does not necessarily denote the same constant in different occurrences. We shall write $C \neq C(L)$ to mean that the constant in question is independent of the parameter L.

2.1 A Characterisation Theorem

In order to formulate our main result, we need the following important theorem which was stated in [8] without proof:

Theorem 2.1

Let $W \in \mathcal{E}$*,* $0 < \alpha < r$*,* $0 < p \leq \infty$*,* $f \in L_{p,W}(\mathbb{R})$ *and assume* (1.6)*.*

Then the following are equivalent:

(a)

$$E_n[f]_{W,p} = O\left(\frac{a_n}{n}\right)^{\alpha}, n \longrightarrow \infty. \tag{2.1}$$

(b)

$$\omega_{r,p}(f, W, t) = O(t^{\alpha}), t \longrightarrow 0^{+}. \tag{2.2}$$

Under more restrictive conditions on W, this was established in [9, pp.185-186] and may be proved using the methods of [1, Corollary 1.6]. For our purposes, it is more important to observe that Theorem 2.1 is not suitable for characterizing optimal orders of smoothness, i.e., it does not include the important case $\alpha = r$. To this end, we replace (2.1) by a different characterisation and prove:

Theorem 2.2

Let $W \in \mathcal{E}$*,* $1 \leq p \leq \infty$ *and* $f \in L_{p,W}(\mathbb{R})$*. Suppose further that*

$$\|P_n^{*(r)} W\|_{L_p(\mathbb{R})} \leq C_1 \left(\frac{n}{a_n}\right)^{r} \psi\left(\frac{a_n}{n}\right), n \longrightarrow \infty \tag{2.3}$$

for some quasi-increasing

$$\psi : [0, \infty] \longrightarrow [0, \infty]$$

satisfying

$$\psi(x) \longrightarrow 0, x \longrightarrow 0^{+}.$$

Then the following hold:

(i)

$$E_n[f]_{W,p} \leq C_2 \left(\int_0^{C_3 \frac{a_n}{n}} \frac{\psi(\tau)}{\tau} d\tau\right), n \longrightarrow \infty. \tag{2.4}$$

(ii)

$$\omega_{r,p}(f, W, t) \leq C_4 \left(\int_0^{C_5 t} \frac{\psi(\tau)}{\tau} d\tau\right), t \longrightarrow 0^{+}. \tag{2.5}$$

Here the $C_j, j = 1, 2, 3, 4, 5$ *are positive and independent of* t *and* n*.*

In particular, if ψ *satisfies for some positive constant* C_6

$$\int_0^{C_6 t} \frac{\psi(\tau)}{\tau} d\tau = O(\psi(t)), t \longrightarrow 0^+$$

then there exist $C_j > 0, j = 7, 8$ *independent of* t *and* n *such that*

$$E_n[f]_{W,p} = O\left(\psi\left(C_7 \frac{a_n}{n}\right)\right), n \longrightarrow \infty \tag{2.6}$$

and

$$\omega_{r,p}(f, W, t) = O(\psi(C_8 t)), t \longrightarrow 0^+. \tag{2.7}$$

We deduce the following analogue of Theorem 2.1.

Theorem 2.3-Characterisation Theorem

Let $W \in \mathcal{E}$, $0 < \alpha \leq r$, $1 \leq p \leq \infty$ *and* $f \in L_{p,W}(\mathbb{R})$.

(a) *Then the following are equivalent:*

$$\omega_{r,p}(f, W, t) = O(t^\alpha), t \longrightarrow 0^+. \tag{2.8}$$

$$\|P_n^{*(r)} W\|_{L_p(\mathbb{R})} = O\left(\frac{n}{a_n}\right)^{r-\alpha}, n \longrightarrow \infty. \tag{2.9}$$

(b) *In particular, the following are equivalent:*

$$\omega_{r,p}(f, W, t) = O(t^r), t \longrightarrow 0^+. \tag{2.10}$$

$$\|P_n^{*(r)} W\|_{L_p(\mathbb{R})} = O(1), n \longrightarrow \infty. \tag{2.11}$$

Remark 2.4

(a) We believe that is unlikely that (2.1) and (2.2) should hold with $\alpha = r$. Indeed it seems that the characterisation (2.9) is the better replacement. We deduce that in the range for which $\omega_{r,p}(f, W, ;)$ and $\omega_{r+1,p}(f, W, ;)$ have different behaviour, $E_n[f]_{W,p}$ yields information on $\omega_{r+1,p}(f, W, ;)$ and $\|P_n^{*(j)} W\|_{L_p(\mathbb{R})}$ yields information on $\omega_{j,p}(f, W, ;)$ for $j = r$ and $j = r + 1$.

(b) Concerning the relationship between $\omega_{r,p}(f, W, ;)$ and $\omega_{r+1,p}(f, W, ;)$ a Marchaud inequality was proved in [8].

We now establish:

Theorem 2.5-Quasi r-Monotonicity of the modulus

Let $W \in \mathcal{E}$, $0 < p \leq \infty$, $f \in L_{p,W}(\mathbb{R})$, $t \in (0, D)$, $r \geq 1$ and assume (1.6). *Then there exists $C_1 > 0$ independent of f and t such that*

$$\omega_{r+1,p}(f, W, t) \leq C_1 \omega_{r,p}(f, W, t). \tag{2.12}$$

Finally we are able to prove:

Theorem 2.6-Existence Theorem for Derivatives

Let $W \in \mathcal{E}$, $0 < p \leq \infty$, $f \in L_{p,W}(\mathbb{R})$, $n \geq n_0$ and $q = \min(1, p)$. Moreover assume (1.6). *Then if for some positive integer k*

$$\sum_{j=1}^{\infty} \left(\frac{2^{j-1}n}{a_{2^{j-1}n}} \right)^{kq} E_{2^{j-1}n}[f]_{W,p}^{q} < \infty$$

the following hold:

(a)

$$f^{(k)} W \in L_p(\mathbb{R}).$$

(b) For some $C_1 \neq C_1(n)$

$$\begin{aligned} &\|(f - P_n^*)^{(k)} W\|_{L_p(\mathbb{R})} \\ &\leq C_1 \left(\sum_{j=1}^{\infty} \left(\frac{2^{j-1}n}{a_{2^{j-1}n}} \right)^{kq} E_{2^{j-1}n}[f]_{W,p}^{q} \right)^{\frac{1}{q}}. \end{aligned} \tag{2.13}$$

Remark 2.7

We remark that it is possible under our hypotheses to reformulate all our results for $n \geq r - 1$.

3 Our Proofs

In this section, we present the proofs of Theorems 2.2, 2.3, 2.5 and 2.6.

3.1 Characterisation Theorem

We begin with:

The Proof of Theorem 2.2

We choose a large natural number M and fix it. For the moment we do not specify the size of M as this will be done later in the proof for clarity.

Let $P^*_{Mn}(f) = P^*_{Mn}$ be the best approximant to f from $\mathcal{P}_{Mn}$ satisfying

$$\|(f - P^*_{Mn})W\|_{L_p(\mathbb{R})} = E_{Mn}[f]_{W,p}. \tag{3.1}$$

Moreover let $P^*_n(P^*_{Mn})$ be the best approximant to P^*_{Mn} from $\mathcal{P}_n$ satisfying,

$$\|(P^*_{Mn} - P^*_n(P^*_{Mn}))W\|_{L_p(\mathbb{R})} = E_n[P^*_{Mn}]_{W,p}. \tag{3.2}$$

First observe that using (1.3) and the fact that $P^*_n(P^*_{Mn})$ is a polynomial of degree at most n gives

$$\begin{aligned} E_n[f]_{W,p} &= \inf_{P \in \mathcal{P}_n} \|(f - P)W\|_{L_p(\mathbb{R})} \\ &\leq \|(f - P^*_n(P^*_{Mn}))W\|_{L_p(\mathbb{R})}. \end{aligned} \tag{3.3}$$

Then (3.1) and (3.3) yield

$$\begin{aligned} I_n : &= \|(P^*_{Mn} - P^*_n(P^*_{Mn}))W\|_{L_p(\mathbb{R})} \\ &\geq \|(f - P^*_n(P^*_{Mn}))W\|_{L_p(\mathbb{R})} - \|(f - P^*_{Mn})W\|_{L_p(\mathbb{R})} \\ &\geq E_n[f]_{W,p} - E_{Mn}[f]_{W,p}. \end{aligned} \tag{3.4}$$

Next we need the following estimate of a_u which follows from [8, (2.7)]:

Given $u \geq 1$ and $v \geq v_0$, there exist positive constants α, β, γ and δ depending only on A, B and λ (recall (1.1)) such that

$$\delta u^{1/1+\beta} \leq \frac{a_{uv}}{a_v} \leq \gamma u^{1/1+\alpha}. \tag{3.5}$$

Then using (1.7), (3.2), (1.14), (2.3) and (3.5) we have

$$\begin{aligned} I_n &\leq C_1 \omega_{r,p} \left(P^*_{Mn}, W, \frac{a_n}{n} \right) \\ &\leq C_2 \psi(\frac{a_{Mn}}{Mn}). \end{aligned} \tag{3.6}$$

Here, $C_2 \neq C_2(n)$.

The estimates (3.4) and (3.6) then readily give

$$\begin{aligned} E_n[f]_{W,p} &\leq C_3 \sum_{k=0}^{\infty} I_{M^k n} \\ &\leq C_4 \sum_{k=1}^{\infty} \psi\left(\frac{a_{M^k n}}{M^k n}\right) \\ &= C_4 S_n \end{aligned} \tag{3.7}$$

where

$$S_n := \sum_{k=1}^{\infty} \psi\left(\frac{a_{M^k n}}{M^k n}\right), n \geq 1 \tag{3.8}$$

and $C_4 \neq C_4(n)$.

We now estimate (3.8) in terms of an integral.

Using (3.5) and recalling that γ and α were independent of u and v, we choose M at the start of the proof so large that

$$M > \exp\left(\frac{1+\alpha}{\alpha}\right)\gamma^{\frac{1+\alpha}{\alpha}}.$$

(3.5) then shows that there exists n_0 such that uniformly for $k \geq 1$ and $n \geq n_0$,

$$\int_{\frac{a_{M^k n}}{M^k n}}^{\frac{a_{M^{k-1} n}}{M^{k-1} n}} \frac{1}{\tau} d\tau \geq 1.$$

The quasi-monotonicity of ψ then yields,

$$\begin{aligned} S_n &\leq C_5 \sum_{k=1}^{\infty} \int_{\frac{a_{M^k n}}{M^k n}}^{\frac{a_{M^{k-1} n}}{M^{k-1} n}} \frac{\psi(\tau) d\tau}{\tau} \\ &\leq C_6 \int_0^{\frac{a_n}{n}} \frac{\psi(\tau)}{\tau} d\tau \end{aligned} \tag{3.9}$$

where $C_6 \neq C_6(n)$.

Substituting (3.9) into (3.7) gives (2.4).

Now let $0 < t < D$ and define $n := n(t)$ by (1.9).

Then using (1.10), (1.11), (1.8), (1.13), (2.3) and (3.7), we proceed much as in the proof of (2.4) and obtain

$$\begin{aligned}\omega_{r,p}(f,W,t) &\leq C_1\omega_{r,p}\left(f,W,\frac{a_{Mn}}{Mn}\right)\\ &\leq C_2K_{r,p}\left(f,W,\left(\frac{a_{Mn}}{Mn}\right)^r\right)\\ &\leq C_3\left(\|(f-P^*_{Mn})W\|_{L_p(\mathbb{R})}+\left(\frac{a_{Mn}}{Mn}\right)^r\|P^{*(r)}_{Mn}W\|_{L_p(\mathbb{R})}\right)\\ &\leq C_4\left(E_{Mn}[f]_{W,p}+\psi\left(\frac{a_{Mn}}{Mn}\right)\right)\\ &\leq C_5\left(\sum_{k=0}^{\infty}\psi\left(\frac{a_{M^{k+1}n}}{M^{k+1}n}\right)\right)\leq C_6\int_0^{C_7t}\frac{\psi(\tau)}{\tau}d\tau. \qquad (3.10)\end{aligned}$$

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Here C_6 and C_7 are independent of t. Thus we have (2.5). (2.6) and (2.7) then follow easily. □

We may proceed with

The Proof of Theorem 2.3

We apply Theorem 2.2 with $\psi(\tau):=\tau^\alpha$. This then shows that (2.9) implies (2.8). The other way follows from (1.10), (1.11) and (1.12). The equivalence of (2.10) and (2.11) follow from part (a) of Theorem 2.2 by setting $\alpha=r$. □

We now present:

The Proof of Theorem 2.5

Let $q=\min(1,p)$ and let P^*_n be the best approximant to f satisfying (1.13). Then (1.10), (1.13), (1.6), (1.7) and (1.12) give for $n\geq n_0$,

$$\begin{aligned}&\omega_{r+1,p}\left(f,W,\frac{a_n}{n}\right)^q\\ &\quad\leq C_1\left(\|(f-P^*_n)W\|^q_{L_p(\mathbb{R})}+\left(\frac{a_n}{n}\right)^{(r+1)q}\|P^{*(r+1)}_nW\|^q_{L_p(\mathbb{R})}\right)\\ &\quad\leq C_2\left(E_n[f]^q_{W,p}+\left(\frac{a_n}{n}\right)^{rq}\|P^{*(r)}_nW\|^q_{L_p(\mathbb{R})}\right)\\ &\quad\leq C_3\omega_{r,p}\left(f,W,\frac{a_n}{n}\right)^q. \qquad (3.11)\end{aligned}$$

Here $C_3\neq C_3(f,n)$.

Now let $0<t<D$ and determine $n:=n(t)$ by (1.9) Then (3.11) and (1.11) together imply (2.12). □

We finish this section with

The Proof of Theorem 2.6

Let P_n^* be the best approximant to f satisfying (1.13). Then much as in [3, Theorem 2.3] we write for a.e $x \in \mathbb{R}$,

$$f(x) = P_n^*(x) + \sum_{j=1}^{\infty}(P_{2^j n}^*(x) - P_{2^{j-1}n}^*(x)). \tag{3.12}$$

Now apply (3.12) together with (1.6). This gives,

$$\begin{aligned} &\|(f - P_n^*)^{(k)}W\|_{L_p(\mathbb{R})}^q \\ \leq\ & C_1 \sum_{j=1}^{\infty} \left(\frac{2^j n}{a_{2^j n}}\right)^{kq} \|(P_{2^j n}^* - P_{2^{j-1}n}^*)W\|_{L_p(\mathbb{R})}^q \\ \leq\ & C_2 \sum_{j=1}^{\infty} \left(\frac{2^{j-1} n}{a_{2^{j-1} n}}\right)^{kq} E_{2^{j-1}n}^q[f]_{W,p}. \end{aligned}$$

Here, $C_2 \neq C_2(n, f)$. Taking qth roots gives the theorem. □

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