The Lebesgue function and Lebesgue constant of Lagrange Interpolation for Erdős Weights

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Abstract

We establish pointwise as well as uniform estimates for Lebesgue functions associated with a large class of Erdős weights on the real line. An Erdős weight is of the form:

$$W := \exp(-Q)$$

where $Q : \mathbb{R} \to \mathbb{R}$ is even and is of faster than polynomial growth at infinity. The archetypal examples are

(i)

$$W_{k,\alpha}(x) := \exp\left(-Q_{k,\alpha}(x)\right)$$

where

$$Q_{k,\alpha}(x) := \exp_k\left(|x|^{\alpha}\right), \alpha > 1,\ k \geq 1.$$

Here $\exp_k := \exp\left(\exp\left(\exp(...)\right)\right)$ denotes the kth iterated exponential.

(ii)

$$W_{A,B}(x) := \exp\left(-Q_{A,B}(x)\right)$$

where

$$Q_{A,B}(x) := \exp\left(\log\left(A + x^2\right)\right)^{B}, B > 1 \text{ and } A > A_0.$$

For a carefully chosen system of nodes

$$\chi_n := \{\xi_1, \xi_2, ..., \xi_n\}, n \geq 1,$$

our results imply in particular, that the Lebesgue constant

$$\|\Lambda_n (W_{k,\alpha}, \chi_n)\|_{L_\infty(\mathbb{R})} := \sup_{x\in\mathbb{R}} |\Lambda_n (W_{k,\alpha}, \chi_n)| (x)$$

satisfies uniformly for $n \geq N_0$,

$$\|\Lambda_n (W_{k,\alpha}, \chi_n)\|_{L_\infty(\mathbb{R})} \sim \log n.$$

Moreover, we show that this choice of nodes is optimal with respect to the zeros of the orthonormal polynomials generated by W^2. Indeed, let

$$U_n := \{x_{j,n} : 1 \leq j \leq n\}, n \geq 1,$$

where the $x_{k,n}$ are the zeros of the orthogonal polynomials $p_n (W^2, .)$ generated by W^2. Then in particular, we have uniformly for $n \geq N$,

$$\|\Lambda_n (W_{k,\alpha}, U_n)\|_{L_\infty(\mathbb{R})} \sim n^{\frac{1}{6}} \left(\prod_{j=1}^{k} \log_j n\right)^{\frac{1}{6}}.$$

Here, $\log_j := \log(\log(\log(...)))$ denotes the jth iterated logarithm.

We deduce sharp theorems of uniform convergence of weighted Lagrange interpolation together with rates of convergence. In particular, these results apply to $W_{k,\alpha}$ and $W_{A,B}$.

1 Introduction and Statement of Results

In this paper, we investigate Lebesgue bounds and uniform convergence of Lagrange interpolation for Erdős weights. We recall that an Erdős weight has the form:

$$W := \exp(-Q)$$

where $Q : \mathbb{R} \to \mathbb{R}$ is even and is of faster than polynomial growth at infinity. The archetypal examples are

(i)

$$W_{k,\alpha}(x) := \exp(-Q_{k,\alpha}(x)), \tag{1.1}$$

where

$$Q_{k,\alpha}(x) := \exp_k\left(|x|^\alpha\right),\ k \geq 1, \alpha > 1.$$

Here $\exp_k := \exp(\exp(\exp(...)))$ denotes the kth iterated exponential.

(ii)

$$W_{A,\beta}(x) := \exp\left(-Q_{A,B}(x)\right) \tag{1.2}$$

where

$$Q_{A,B}(x) := \exp\left(\log\left(A + x^2\right)\right)^B, B \geq 1 \text{ and } A \text{ is large enough but fixed.}$$

Throughout, let $f : \mathbb{R} \to \mathbb{R}$ be continuous and satisfy the decay condition,

$$\lim_{|x| \longrightarrow \infty} |fW|(x) = 0. \tag{1.3}$$

We set

$$E_n[f]_{W,\infty} := \inf_{P \in \mathcal{P}_n} \|(f-P)(x) W(x)\|_{L_\infty(\mathbb{R})} \tag{1.4}$$

to be the error of best weighted polynomial approximation to f from $\mathcal{P}_n, n \geq 1$.

Here, $\mathcal{P}_n$ denotes the class of polynomials of degree $\leq n$.

It is well known ([9]) that

$$E_n[f]_{W,\infty} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Now let

$$\chi_n := \{\xi_1, \xi_2, ..., \xi_n\}, n \geq 1,$$

be an arbitrary set of nodes. The Lagrange interpolation polynomial to f with respect to χ_n is denoted by $L_n[f, W, \chi_n]$. Thus, if

$$l_{j,n}(\chi_n) \in \mathcal{P}_{n-1}, 1 \leq j \leq n,$$

are the fundamental polynomials of Lagrange interpolation at $\xi_j, 1 \leq j \leq n$, satisfying,

$$l_{j,n}(\chi_n)(\xi_{j,n}) = \delta_{j,k},\ 1 \leq k \leq n,$$

then,

$$L_n[f, W, \chi_n](x) = \sum_{j=1}^{n} f(\xi_{j,n})\, l_{j,n}(\chi_n)(x) \in \mathcal{P}_{n-1}. \tag{1.5}$$

Now write,

$$
\begin{aligned}
&\|W\,(f-L_n[f,W,\chi_n])\|_{L_\infty(\mathbb{R})}\\
&\quad\le E_{n-1}[f]_{W,\infty}\left(1+\left\|W(x)\sum_{j=1}^{n}|l_{j,n}(\chi_n)(x)|\,W^{-1}(\xi_j)\right\|_{L_\infty(\mathbb{R})}\right)\\
&\quad= E_{n-1}[f]_{W,\infty}\left(1+\|\Lambda_n(W,\chi_n)\|_{L_\infty(\mathbb{R})}\right) \qquad (1.6)
\end{aligned}
$$

where $\|\Lambda_n(W,\chi_n)\|_{L_\infty(\mathbb{R})}$ is called the Lebesgue constant with respect to the weight W and the set of nodes χ_n, and $\Lambda_n(W,\chi_n)$ is the corresponding Lebesgue function.

Using (1.6), we see that estimates of the size of the Lebesgue constant enable one to deduce theorems on uniform convergence of Lagrange interpolation. As the subject of weighted Lagrange interpolation is an extensively researched and widely studied subject, we refer the reader to [1, 5, 6, 7, 10, 11, 12, 13, 14, 15].

Now given a weight $W:\mathbb{R}\longrightarrow(0,1]$ as above, we may define orthonormal polynomials

$$p_n(x):=p_n(W^2,x)=\gamma_n x^n+...,\ \text{with } \gamma_n=\gamma_n(W^2)>0,$$

satisfying

$$\int_{\mathbb{R}} p_n(W^2,x)p_m(W^2,x)W^2(x)dx=\delta_{mn}.$$

We denote the zeros of p_n by

$$-\infty<x_{n,n}<x_{n-1,n}<...<x_{2,n}<x_{1,n}<\infty.$$

Put

$$U_n:=\{x_{j,n}:1\le j\le n\}, n\ge 1. \qquad (1.7)$$

To formulate our results, we need a suitable class of Erdős weights from [8] .

Definition 1.1. Let $W:=\exp(-Q)$, where $Q:\mathbb{R}\to\mathbb{R}$ is even, continuous, Q'' exists in $(0,\infty)$, $Q^{(j)}\ge 0$ in $(0,\infty)$, $j=0,2,Q^{(1)}>0$ in $(0,\infty)$ and the function

$$T(x):=1+\frac{xQ''(x)}{Q'(x)} \qquad (1.8)$$

is increasing in $(0, \infty)$ with

$$\lim_{x \longrightarrow \infty} T(x) = \infty, \ T\left(0^+\right) := \lim_{x \longrightarrow 0^+} T(x) > 1. \tag{1.9}$$

Moreover, we assume that for some $C_j > 0$, $1 \leq j \leq 3$,

$$C_1 \leq \frac{T(x)}{\frac{xQ'(x)}{Q(x)}} \leq C_2, \ x \geq C_3 \tag{1.10}$$

and for every $\varepsilon > 0$,

$$T(x) = O\left((Q(x))^{\varepsilon}\right), \ x \longrightarrow \infty. \tag{1.11}$$

Then, we write $W \in \mathcal{E}$.

The principle examples of $W \in \mathcal{E}$ are $W_{k,\alpha}$ and $W_{A,B}$ given by (1.1) and (1.2) respectively. For more on this subject we refer the reader to [2, 3, 4, 8].

To state our results, we need some more notation:
We need the Mhaskar-Rakhmanov-Saff number a_u defined as the positive root of the equation

$$u = \frac{2}{\pi} \int_0^1 \frac{a_u t Q'(a_u t)\, dt}{\sqrt{1-t^2}}, u > 0. \tag{1.12}$$

Here, a_u exists and is a strictly increasing function of u $[8, 9]$. Amongst its uses is the infinite-finite range inequality

$$\|PW\|_{L_\infty(\mathbb{R})} = \|PW\|_{L_\infty[-a_n, a_n]}, \ P \in \mathcal{P}_n, \tag{1.13}$$

Note that a_n depends only on the degree of the polynomial P and not on P itself.

Now choose $y_0 \in [-a_n, a_n]$ so that

$$|p_n W(y_0)| = \|p_n W\|_{L_\infty(\mathbb{R})}. \tag{1.14}$$

As W is even, we may assume that $y_0 \geq 0$. We will show later that in fact $y_0 > 0$ and is very "close" to a_n. Fix y_0 as above.

Finally set

$$\delta_n := (nT(a_n))^{\frac{-2}{3}}, n \geq 1, \tag{1.15}$$

and

$$\Psi_n(x) := \begin{cases} \max\left\{\sqrt{1-\frac{|x|}{a_n}+L\delta_n}, \frac{1}{T(a_n)\sqrt{1-\frac{|x|}{a_n}+L\delta_n}}\right\} & ,|x| \le a_n \\ \Psi(a_n) & ,|x| \ge a_n \end{cases}. \tag{1.16}$$

Here, $L > 0$ is fixed, but large enough throughout.

For more on these special sequences of functions, we refer the reader to [5, 8].

Here and throughout,

$$a_n = O(b_n), a_n \sim b_n \text{ and } a_n = o(b_n)$$

will mean respectively that there exist constants $C_j > 0$, $j = 1, 2, 3$, independent of n, such that

$$\frac{a_n}{b_n} \le C_1, C_2 \le \frac{a_n}{b_n} \le C_3 \text{ and } \lim_{n\longrightarrow\infty}\left|\frac{a_n}{b_n}\right| = 0.$$

Similar notation will be used for functions and sequences of functions.

Bounds for Lebesgue constants and uniform convergence of Lagrange Interpolation for $U_n, n \ge 1$.

We begin our investigation with the sequence of nodes, $U_n, n \ge 1$, defined by (1.7).

We prove:

Theorem 1.2. Let $W \in \mathcal{E}$. Then, uniformly for $n \ge N_0$,

$$\|\Lambda_n(W, U_n)\|_{L_\infty(\mathbb{R})} \sim n^{\frac{1}{6}} T(a_n)^{\frac{1}{6}}. \tag{1.17}$$

In particular, given $\varepsilon > 0$, there exists $C > 0$ independent of n such that

$$\|\Lambda_n(W, U_n)\|_{L_\infty(\mathbb{R})} \le C n^{\frac{1}{6}+\varepsilon}.$$

We deduce:

Corollary 1.3. Let $W \in \mathcal{E}$ and $r \geq 1$. Then there exists $C_j > 0$ $j = 1, 2$ independent of n and f so that for $n \geq N_0$,

(a)

$$\begin{aligned} &\|(f - L_n[f, W, U_n])\, W\|_{L_\infty(\mathbb{R})} \\ &\leq C_1 E_{n-1}[f]_{W,\infty} n^{\frac{1}{6}} T\,(a_n)^{\frac{1}{6}} \\ &\leq C_2 \omega_{r,\infty}(f, W, \frac{a_n}{n}) n^{\frac{1}{6}} T\,(a_n)^{\frac{1}{6}}\,. \end{aligned} \tag{1.18}$$

Here,

$$\omega_{r,\infty}(f, W, t) :=$$
$$\left[\sup_{0<h\leq t} \left\| W \Delta^r_{h\Phi_t^{\frac{1}{2}}(x)}(f) \right\|_{L_\infty(|x|\leq\sigma(2t))} + \inf_{P\in\mathcal{P}_{r-1}} \|(f - P)\, W\|_{L_\infty(|x|\geq\sigma(4t))}\right], t > 0$$

is the weighted modulus of smoothness of f ,

$$\sigma\,(t) := \inf\left\{a_u : \frac{a_u}{u} \leq t\right\}, \tag{1.19}$$

$$\Phi_t(x) := \left|1 - \frac{|x|}{\sigma\,(t)}\right| + T\,(\sigma\,(t))^{-1}\,,\ x \in \mathbb{R}, \tag{1.20}$$

and for an interval J and $h > 0$,

$$\Delta^r_h(f, x, J) := \left\{ \begin{array}{ll} \sum_{i=0}^r \left(\begin{array}{c} r \\ i \end{array}\right) (-1)^i f\left(x + \frac{rh}{2} - ih\right) & , x \pm \frac{rh}{2} \in J \\ 0 & , \text{otherwise} \end{array} \right\}.$$

(b) Moreover, if f satisfies $f^{(r)}W \in L_\infty(\mathbb{R})$, then given $\varepsilon > 0$,

$$\begin{aligned} &\|(f - L_n[f, W, U_n])\, W\|_{L_\infty(\mathbb{R})} \\ &\leq C_3 \left(\frac{a_n}{n}\right)^r n^{\frac{1}{6}} T\,(a_n)^{\frac{1}{6}} \qquad (1.21) \\ &\leq C_3 n^{\frac{1}{6}+\varepsilon-r}. \qquad (1.22) \end{aligned}$$

Here $C_3 > 0$ is independent of n.

Thus we can ensure uniform convergence for every $r \geq 1$.

Remark.

It is instructive at this point to recall that for $Q = Q_{k,\alpha}$ of (1.1),

$$T(a_n) = \prod_{j=1}^{k} \log_j n.$$

Moreover, in general, given $\varepsilon > 0$ and $n \geq 1$,

$$T(a_n) = O(n^{\varepsilon}).$$

(See also (2.7)). We thus observe that we may dispense with the $T(a_n)^{\frac{1}{6}}$ on the right hand side of (1.17) by inserting an extra weighting factor into the left hand side of (1.17) in the following sense:

Under the hypotheses of Theorem 1.2, we have uniformly for $n \geq N_0$,

$$\left\| \Lambda_n(W, U_n) \left(\left| 1 - \frac{|x|}{a_n} \right| + T(a_n)^{-1} \right)^{\frac{1}{6}} \right\|_{L_\infty(\mathbb{R})} \sim n^{\frac{1}{6}}. \tag{1.23}$$

This follows easily using the proof of (1.17) and (2.11).

A better behaving Lebesgue function.

We observe that although (1.21) yields uniform convergence for every $r \geq 1$, we can substantially improve our results, by choosing our interpolation points more carefully. For weights on the real line, J. Szabados was the first to exploit this idea and many of the proofs in this section rely heavily on his ideas $[14]$. Motivated by (1.13) and recalling the definition of y_0 in (1.14) and U_n in (1.7), we set:

$$V_{n+2} := \{-y_0, y_0\} \cup U_n, \ n \geq 1,$$

and prove:

Theorem 1.4. Let $W \in \mathcal{E}$. Then uniformly for $n \geq N_0$,

$$\|\Lambda_{n+2}(W, V_{n+2})\|_{L_\infty(\mathbb{R})} \sim \log n. \tag{1.24}$$

Thus, by adding two completely new points of interpolation, we can achieve the much better order $\log n$ in comparison to the order $(nT(a_n))^{\frac{1}{6}}$ that we obtained merely using the zeros of p_n.

We deduce,

Corollary 1.5. Let $W \in \mathcal{E}$ and $r \geq 1$. Then there exists $C_j > 0$ $j = 1, 2$ independent of f and n so that for $n \geq N_0$,

(a)

$$\begin{aligned} &\|(f - L_{n+1}[f, W, V_{n+2}])\, W\|_{L_\infty(\mathbb{R})} \\ &\leq C_1 E_n[f]_{W,\infty} \log n \\ &\leq C_2 \omega_{r,\infty}(f, W, \frac{a_n}{n}) \log n. \end{aligned} \tag{1.25}$$

(b) Moreover, if f satisfies $f^{(r)}W \in L_\infty(\mathbb{R})$ then, given $\varepsilon > 0$,

$$\|(f - L_n[f, W, U_n])\, W\|_{L_\infty(\mathbb{R})}$$

$$\leq C_3 \left(\frac{a_n}{n}\right)^r \log n \tag{1.26}$$

$$\leq C_3 n^{-r+\varepsilon} \log n. \tag{1.27}$$

Here $C_3 > 0$ is independent of n.

Remark.

A natural question arises as to whether (1.24) holds (in a lower bound sense) for any system of nodes, at least for some Erdős weight. This and related questions will be considered in a future paper.

Pointwise estimates for $\Lambda_n(W, U_n)$.

We present pointwise estimates for $\Lambda_n(W, U_n)$. We emphasize our results and briefly sketch their proofs in Section 5 as the arguments are straightforward, but rather lengthy.

Theorem 1.6. Let $W \in \mathcal{E}$.

(a) Then for $n \geq N_0$,there exists $C > 0$ such that for $|x| \leq a_n \left(1 + \frac{L}{2}\delta_n\right)$,

$$\begin{aligned} \Lambda_n(W, U_n)(x) \quad \leq \quad & C\left[1 + \sqrt{a_n}\,|p_n W|(x)\right. \\ & \times \left.\left[\left(1 - \frac{|x|}{a_n} + L\delta_n\right)^{\frac{1}{4}} \log\left(\frac{n\left(1 - \frac{|x|}{a_n} + L\delta_n\right)}{\Psi_n(x)}\right) + 1\right]\right]. \end{aligned} \tag{1.28}$$

Moreover, we have uniformly for $|x| \leq x_{1,n}$ and n,

$$\Lambda_n(W, U_n)(x) \sim 1 + \sqrt{a_n} \left|p_n W\right|(x) \times \left[\left(1 - \frac{|x|}{a_n} + L\delta_n\right)^{\frac{1}{4}} \log\left(\frac{n\left(1 - \frac{|x|}{a_n} + L\delta_n\right)}{\Psi_n(x)}\right) + 1 \right]. \tag{1.29}$$

(b) Uniformly for $n \geq N_0$ and $a_n(1 + \frac{L}{2}\delta_n) \leq |x| \leq 2a_n$,

$$\Lambda_n(W, U_n)(x) \sim \sqrt{a_n} \left|p_n W\right|(x)[1 + \delta_n^{\frac{1}{4}}]. \tag{1.30}$$

(c) Uniformly for $n \geq N_0$ and $|x| \geq 2a_n$,

$$\Lambda_n(W, U_n)(x) \sim \frac{a_n^{\frac{3}{2}} \left|p_n W\right|(x)}{|x|}[1 + \delta_n^{\frac{1}{4}}]. \tag{1.31}$$

Structure of this paper.

We close this section with some notation and remarks concerning the structure of this paper. Throughout, $C, C_1, C_2... > 0$ will denote constants independent of n, x and $P \in \mathcal{P}_n$. The same symbol does not necessarily denote the same constant in different occurrences. We write $C \neq C(L)$ to indicate that C is independent of L.

This paper is organized as follows:

In Section 2, we present our technical lemmas. In Section 3, we present the proofs of our upper bounds for (1.17) and (1.24). In Section 4, we prove Theorems 1.2 and 1.4 and Corollaries 1.3 and 1.5. Finally in Section 5, we sketch briefly the main ideas in the proof of Theorem 1.6.

2 Technical Lemmas

Lemma 2.1. Let $W \in \mathcal{E}$ and set

$$x_{0,n} := x_{1,n}(1 + L\delta_n) \text{ and } x_{n,n+1} := -x_{0,n}.$$

(a) There exists $A > 0$ independent of n and L such that for $n \geq 1$,

$$\left|\frac{x_{1,n}}{a_n} - 1\right| \leq A\delta_n. \tag{2.1}$$

(b) Uniformly for $n \geq 2$ and $0 \leq j \leq n-1$,

$$x_{j,n} - x_{j+1,n} \sim \frac{a_n}{n} \Psi_n \left(x_{j,n} \right). \tag{2.2}$$

(c) Uniformly for $n \geq 2$ and $0 < j \leq n-1$,

$$1 - \frac{|x_{j,n}|}{a_n} + L\delta_n \sim 1 - \frac{|x_{j+1,n}|}{a_n} + L\delta_n \tag{2.3}$$

and

$$\Psi_n \left(x_{j,n} \right) \sim \Psi_n \left(x_{j+1,n} \right). \tag{2.4}$$

(d) For $n \geq 1$,

$$\sup_{x \in \mathbb{R}} |p_n W| (x) \left| 1 - \frac{|x|}{a_n} \right|^{\frac{1}{4}} \sim a_n^{\frac{-1}{2}} \tag{2.5}$$

and

$$\sup_{x \in \mathbb{R}} |p_n W| (x) \sim n^{\frac{1}{6}} T \left(a_n \right)^{\frac{1}{6}} a_n^{\frac{-1}{2}}. \tag{2.6}$$

Proof. This is part of Lemma 2.1 of [5] .□

Now fix A in (2.1) .

Lemma 2.2. Let $W \in \mathcal{E}$.

(a) Given $\varepsilon > 0$ and $n \geq 1$, there exists $C > 0$ independent of n such that,

$$a_n \leq C n^{\varepsilon}, \quad T \left(a_n \right) \leq C n^{\varepsilon} \text{ and } \delta_n \leq C T \left(a_n \right)^{-\varepsilon}. \tag{2.7}$$

(b) Given $0 < \alpha < \beta$, we have uniformly for $n \geq C$,

$$T \left(a_{\alpha n} \right) \sim T \left(a_{\beta n} \right). \tag{2.8}$$

(c) Uniformly for $u \in (C, \infty), v \in \left[\frac{u}{2}, 2u \right]$, we have

$$\left| \frac{a_u}{a_v} - 1 \right| \sim \left| \frac{u}{v} - 1 \right| T \left(a_n \right)^{-1}. \tag{2.9}$$

(d) Given $m \in \mathbb{N}$ and $n \geq N_0$, we have for every $\{P_k\}_{k=1}^{m} \in \mathcal{P}_n$

$$\left\| W \sum_{k=1}^{m} |P_k| \right\|_{L_\infty(\mathbb{R})} = \left\| W \sum_{k=1}^{m} |P_k| \right\|_{L_\infty[-a_n, a_n]}. \tag{2.10}$$

Moreover, given $r > 1$, there exists $C = C(r) > 0$ independent of n, m and P_k such that

$$\left\| W \left(\left| 1 - \frac{|x|}{a_n} \right| + T(a_n)^{-1} \right)^{\frac{1}{6}} \sum_{k=1}^{m} |P_k| \right\|_{L_\infty(\mathbb{R})}$$
$$\leq C \left\| W \left(\left| 1 - \frac{|x|}{a_n} \right| + T(a_n)^{-1} \right)^{\frac{1}{6}} \sum_{k=1}^{m} |P_k| \right\|_{L_\infty[-a_{r(n+1)}, a_{r(n+1)}]} . \quad (2.11)$$

Proof. (a)–(c) are part of Lemma 2.3 of [5], (2.10) follows as in Lemma 1 of [14] and then (2.11) follows using (2.10) and the method of Lemma 3.3 in [3]. □

Our next lemma establishes how "close" y_0 is to a_n.

Lemma 2.3. Let $W \in \mathcal{E}, n \geq N_0$ and y_0 as in (1.14). Then, we have

$$a_n(1 - B\delta_n) \leq y_0 \leq a_n \quad (2.12)$$

for some $B > 0$ independent of n and L.

Proof. By (2.5), (2.6) and the definition of δ_n [see (1.15)], there exist $C_j > 0, j = 1, 2$ such that

$$C_1 a_n^{\frac{-1}{2}} (nT(a_n))^{\frac{1}{6}} \leq |p_n(y_0)| W(y_0)$$
$$\leq C_2 a_n^{\frac{-1}{2}} \min \left\{ \left| 1 - \frac{y_0}{a_n} \right|^{\frac{-1}{4}}, \delta_n^{\frac{-1}{4}} \right\}. \quad (2.13)$$

Then, this gives

$$\max \left\{ \left| 1 - \frac{y_0}{a_n} \right|, \delta_n \right\} \leq C_3 \delta_n. \quad (2.14)$$

Now by the definition of y_0, we have clearly that $y_0 \leq a_n$. Moreover, if $y_0 \geq a_n(1 - \delta_n)$ then (2.12) is satisfied with $B = 1$. Suppose then, that

$$0 \leq y_0 < a_n(1 - \delta_n).$$

Then (2.14) becomes

$$\left(1 - \frac{y_0}{a_n} \right) \leq C_4 \delta_n$$

which again implies (2.12) with $B = C_4$.□

Now, fix B in (2.12).

Lemma 2.4. Let $W \in \mathcal{E}$.

(a) Uniformly for $n \geq 1$, $1 \leq j \leq n$ and $x \in \mathbb{R}$,

$$|l_{j,n}(U_n)(x)| \sim \frac{a_n^{\frac{3}{2}}}{n} \Psi_n W\left(x_{j,n}\right) \left(1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right)^{\frac{1}{4}} \left|\frac{p_n\left(x\right)}{x - x_{j,n}}\right|. \qquad (2.15)$$

(b) There exists $C > 0$ such that uniformly for $n \geq 1$, $1 \leq j \leq n$ and $x \in \mathbb{R}$,

$$\left|l_{j,n}\left(U_n\right)(x) W\left(x\right)\right| W^{-1}\left(x_{j,n}\right) \leq C. \qquad (2.16)$$

(c) Uniformly for $n \geq 1$ and $1 \leq j \leq n$,

$$\frac{a_n^{\frac{3}{2}}}{n} \Psi_n\left(x_{j,n}\right) \left(1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right)^{\frac{1}{2}} \left|p_n' W\right|\left(x_{j,n}\right)$$
$$\sim a_n^{\frac{1}{2}} \left|p_{n-1} W\right|\left(x_{j,n}\right) \sim \left(1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right)^{\frac{1}{4}}. \qquad (2.17)$$

(d) For $n \geq 1, 1 \leq j \leq n$ and $|x| \leq a_n$, there exists $C > 0$ such that

$$\left|p_n(x)\right| W(x) \leq C \frac{n}{a_n^{\frac{3}{2}}} \left[\Psi_n\left(x\right) \Psi_n\left(x_{j,n}\right) \left(1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right)^{\frac{1}{2}}\right]^{\frac{-1}{2}} \times$$
$$\times \left|x - x_{j,n}\right|. \qquad (2.18)$$

Proof. (a), (b) and (c) are $(2.13), (2.14)$ and (2.11) resp in $[5]$. (d) is (10.28) in $[8]$.□

Lemma 2.5. Let $W \in \mathcal{E}$ and let $l_{n+1,n+2}\left(V_{n+2}\right)$ and $l_{n+2,n+2}\left(V_{n+2}\right)$ be respectively the fundamental polynomials of degree $\leq n+1$ at the points y_0 and -y_0. Then there exists $C > 0$ such for all $x \in \mathbb{R}$,

$$\left|l_{n+1,n+2}(V_{n+2})\right|(x)\, W\left(x\right) W^{-1}\left(y_0\right) \leq C \qquad (2.19)$$

and

$$\left|l_{n+2,n+2}(V_{n+2})\right|(x)\,W(x)\,W^{-1}(-y_0) \leq C. \tag{2.20}$$

Proof. We prove (2.19). (2.20) is similar. First observe that

$$l_{n+1,n+2}(V_{n+2})(x) = \frac{p_n(x)(y_0+x)}{2y_0 p_n(y_0)} \in \mathcal{P}_{n+1} \tag{2.21}$$

and satisfies

$$l_{n+1,n+2}(V_{n+2})(y_0) = 1, \tag{2.22}$$

$$l_{n+1,n+2}(V_{n+2})(x_{j,n}) = 0, \quad 1 \leq j \leq n \tag{2.23}$$

and

$$l_{n+1,n+2}(V_{n+2})(-y_0) = 0.$$

Observe that by (2.10), we may assume that $|x| \leq a_{n+1}$. Then by (2.6), (2.9), the definition of y_0, (2.12) and (2.21),

$$\begin{aligned}
\left|l_{n+1,n+2}(V_{n+2})W(x)\,W^{-1}(y_0)\right| &\leq C\frac{W(x)\,|p_n(x)|\,|y_0+x|}{2y_0\,|p_n(y_0)|\,W(y_0)} \\
&\leq C_1\frac{a_n^{\frac{-1}{2}}n^{\frac{1}{6}}T(a_n)^{\frac{1}{6}}\,a_n}{2a_n(1-B\delta_n)\,a_n^{\frac{-1}{2}}n^{\frac{1}{6}}T(a_n)^{\frac{1}{6}}} \\
&\leq C_2. \square
\end{aligned}$$

We next need a lemma which gives an estimate of the distance between y_0 and $|x_{j,n}|, 1 \leq j \leq n$.

Lemma 2.6. Let $W \in \mathcal{E}$. Then for $n \geq N_0$ and uniformly for $1 \leq j \leq n$, we have

$$|y_0 - |x_{j,n}|| \sim a_n\left(\left|1 - \frac{|x_{j,n}|}{a_n}\right| + L\delta_n\right). \tag{2.24}$$

Proof. We begin with our lower bound. We consider two cases:

Case 1: $|x_{j,n}| \geq a_n (1 - 2L\delta_n)$.

Note that here,

$$1 - \frac{|x_{j,n}|}{a_n} + L\delta_n \leq 3L\delta_n.$$

Moreover (2.1) implies

$$\left|1 - \frac{|x_{j,n}|}{a_n}\right| + L\delta_n \leq 3L\delta_n \tag{2.25}$$

if L is large enough.

Next observe that by (2.12) and the definition of Ψ_n [see (1.16)], we have that

$$\Psi_n^{\frac{1}{2}}(y_0) \geq \left(T(a_n)^{\frac{1}{2}} \delta_n^{\frac{1}{4}} (B+L)^{\frac{1}{4}}\right)^{-1}. \tag{2.26}$$

Now as Q and $|p_n|$ are both even functions, the definition of Ψ_n (1.16), (2.6), (2.18), (2.25) and (2.26) yield

$$|y_0 - |x_{j,n}|| \geq C_1 a_n \delta_n \geq C_2 a_n \left(\left|1 - \frac{|x_{j,n}|}{a_n}\right| + L\delta_n\right)$$

uniformly for $1 \leq j \leq n$.

Case 2: $|x_{j,n}| \leq a_n (1 - 2L\delta_n)$.

Observe that if L is large enough,

$$|y_0 - |x_{j,n}|| \geq a_n \left(\left|1 - \frac{|x_{j,n}|}{a_n}\right| + L\delta_n\right) - (a_n(1 + L\delta_n) - y_0). \tag{2.27}$$

Now by (2.12),

$$(a_n(1 + L\delta_n) - y_0) \leq \frac{a_n}{2}\left[1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right] \tag{2.28}$$

if

$$1 - \frac{|x_{j,n}|}{a_n} \geq 2\delta_n \left[B + \frac{L}{2}\right]. \tag{2.29}$$

But then it is easy to see that $|x_{j,n}| \leq a_n (1 - 2L\delta_n)$ implies (2.29) if L is large enough and so we have (2.28). (2.27) then becomes

$$|y_0 - |x_{j,n}|| \geq \frac{a_n}{2} \left(\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + L\delta_n \right)$$

and we have our lower bound for this case as well.

The upper bound is easier. We again distinguish two cases:

Case 1: $|x_{j,n}| \leq a_n$.

Here, if L is large enough, we have by (2.12),

$$\begin{aligned} |y_0 - |x_{j,n}|| &\leq La_n\delta_n + a_n \left(1 - \frac{|x_{j,n}|}{a_n}\right) \\ &= a_n \left(\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + L\delta_n \right). \end{aligned}$$

Case 2: $a_n \leq |x_{j,n}| \leq a_n (1 + A\delta_n)$.

Here if L is large enough, we have by (2.1) and (2.12),

$$\begin{aligned} |y_0 - |x_{j,n}|| &\leq Ba_n\delta_n + x_{1,n} - a_n \\ &\leq a_n\delta_n (B + A) \leq a_n \left[\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + L\delta_n \right]. \end{aligned}$$

The lemma is proved.□

Let us put

$$\Delta x_{j,n} := x_{j,n} - x_{j+1,n}\ ,1 \leq j \leq n.$$

We prove:

Lemma 2.7: Let $W \in \mathcal{E}$, $n \geq N_0, r > 1$ and $|x| \leq a_{rn}$. Then there exists $C_j > 0$ $j = 1, 2$ such that for $1 \leq j \leq n$,

(a)

$$\begin{aligned} &W(x)\, l_{j,n}(U_n)(x)\, W^{-1}(x_{j,n}) \\ &\quad \leq C_1 \left(\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + L\delta_n \right)^{\frac{1}{4}} \left(\left| 1 - \frac{|x|}{a_n} \right| + L\delta_n \right)^{\frac{-1}{4}} \frac{\Delta x_{j,n}}{|x - x_{j,n}|}. \quad (2.30) \end{aligned}$$

(b)

$$W(x)\,l_{j,n+2}(V_{n+2})(x)\,W^{-1}(x_{j,n})$$
$$\leq C_1\left(\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n\right)^{\frac{-3}{4}}\left(\left|1-\frac{|x|}{a_n}\right|+L\delta_n\right)^{\frac{3}{4}}\frac{\Delta x_{j,n}}{|x-x_{j,n}|}. \quad (2.31)$$

Proof. We begin first with (2.30). First note that (2.5) and (2.6) show that uniformly for n and x,

$$|p_n(x)|\,W(x)\leq C_1 a_n^{\frac{-1}{2}}\left(\left|1-\frac{|x|}{a_n}\right|+L\delta_n\right)^{\frac{-1}{4}}. \quad (2.32)$$

Then by (2.32),

$$W(x)\,l_{j,n}(U_n)(x)\,W^{-1}(x_{j,n})$$
$$=\frac{W(x)\,|p_n(x)|\,W^{-1}(x_{j,n})}{|p_n'(x_{j,n})|\,|x-x_{j,n}|}$$
$$\leq C_1\frac{a_n^{\frac{-1}{2}}\left(\left|1-\frac{|x|}{a_n}\right|+L\delta_n\right)^{\frac{-1}{4}}W^{-1}(x_{j,n})}{|p_n'(x_{j,n})|\,|x-x_{j,n}|}$$
$$\leq C_2\left(\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n\right)^{\frac{1}{4}}\left(\left|1-\frac{|x|}{a_n}\right|+L\delta_n\right)^{\frac{-1}{4}}\frac{\Delta x_{j,n}}{|x-x_{j,n}|}.$$

by (2.2) and (2.17). So we have (2.30).

We now proceed with (2.31).
First observe that for $1\leq j\leq n$,

$$l_{j,n+2}(V_{n+2})(x)=\left(\frac{y_0^2-x^2}{y_0^2-x_{j,n}^2}\right)l_{j,n}(U_n)(x). \quad (2.33)$$

Next, we claim that

$$|y_0-x|\leq C_3 a_n\left(\left|1-\frac{|x|}{a_n}\right|+L\delta_n\right). \quad (2.34)$$

We consider two cases:

Case 1: $|x| \leq a_n$.

Here much as in the proof of Lemma 2.6,

$$\begin{aligned} |y_0 - |x|| &\leq Ba_n\delta_n + a_n\left(1 - \frac{|x|}{a_n}\right) \\ &\leq C_3 a_n\left(\left|1 - \frac{|x|}{a_n}\right| + L\delta_n\right) \end{aligned}$$

if L is large enough.

Case 2: $a_n < |x| \leq a_{rn}$.
Here, using (2.9) ,

$$\begin{aligned} |x| - a_n &\leq a_{rn} - a_n \\ &\leq C_4 a_n T(a_n)^{-1} \\ &\leq C_5 a_n\left(\left|1 - \frac{|x|}{a_n}\right|\right) \end{aligned}$$

so that

$$\begin{aligned} |y_0 - |x|| &\leq |a_n - y_0| + |a_n - |x|| \\ &\leq C_6 a_n\left(\left|1 - \frac{|x|}{a_n}\right| + L\delta_n\right) \end{aligned}$$

so (2.34) is established. Then (2.24) , (2.30), (2.33) and (2.34) yield (2.31) .□

3 The Proofs of our Upper Bounds

In this section we establish our upper bounds for (1.17) and (1.24) . Throughout we assume that $W \in \mathcal{E}$, $x \in \mathbb{R}$ is fixed and $x_{k(x),n}$ is that zero of p_n closest to x.

We need two lemmas.

Lemma 3.1. There exist M and $\delta > 0$ with the following properties:
(a) If $|x| \in \left[0, a_n\left(1 + \frac{L}{2}\delta_n\right)\right]$ then:

(i)

$$\{j : |j - k(x)| \le 2\} \subseteq \left\{ j : |x - x_{j,n}| \le \frac{Ma_n}{n} \Psi_n(x) \right\}. \tag{3.1}$$

(ii)

$$\left| x - x_{k(x) \pm k, n} \right| \le \delta \frac{a_n}{n} \Psi_n(x), k = 0, 1.$$

(iii)

$$\left| x - x_{k(x) \pm 3, n} \right| > \frac{Ma_n}{n} \Psi_n(x). \tag{3.2}$$

(b) If $|x| \in [a_n(1 + \frac{L}{2}\delta_n), \infty)$,

$$|x - x_{j,n}| > \frac{Ma_n}{n} \Psi_n(x) \tag{3.3}$$

for all $1 \le j \le n$.

Proof. Suppose first that $x \in \left[0, a_n \left(1 + \frac{L}{2}\delta_n\right)\right]$. Observe that if $t \in [x_{j+1,n}, x_{j,n}]$, $1 \le j \le n$, we have

$$\begin{aligned} \left| \frac{1 - \frac{|t|}{a_n} + L\delta_n}{1 - \frac{|x_{j,n}|}{a_n} + L\delta_n} - 1 \right| &\le \frac{1}{a_n} \left| \frac{x_{j,n} - t}{1 - \frac{|x_{j,n}|}{a_n} + L\delta_n} \right| \\ &\le \frac{1}{a_n} \left| \frac{x_{j,n} - x_{j+1,n}}{1 - \frac{|x_{j,n}|}{a_n} + L\delta_n} \right| \le \frac{C\Psi_n(x_{j,n})}{n(L - A)\delta_n} \le \frac{1}{2} \end{aligned} \tag{3.4}$$

by $(1.16), (2.1)$ and (2.2) if L is large enough.

We conclude using (1.16) and (3.4) that

$$\Psi_n(t) \sim \Psi_n(x_{j,n}) \text{ uniformly for } j, n \text{ and } t \in [x_{j+1,n}, x_{j,n}]. \tag{3.5}$$

Now by definition of $x_{k(x),n}$, we must have $x \in \left[x_{k(x)+1,n}, x_{k(x),n}\right]$ or $x \in \left[x_{k(x),n}, x_{k(x)-1,n}\right]$ at least when $x \le x_{1,n}$. Using (2.3) and (2.4) if necessary, we may assume without loss of generality that $x \in \left[x_{k(x)+1,n}, x_{k(x),n}\right]$.

Then by (2.2) and (3.5),

$$\begin{aligned} \left| x - x_{k(x) \pm 2, n} \right| &\le \left| x_{k(x)-2,n} - x_{k(x)+2,n} \right| \\ &\le C \frac{a_n}{n} \Psi_n\left(x_{k(x),n}\right) \\ &\sim \frac{a_n}{n} \Psi_n(x). \end{aligned} \tag{3.6}$$

Using (3.6) and (2.2) we see that it is possible to choose M such that (3.1) holds at least when $x \leq x_{1,n}$. Suppose $x \geq x_{1,n}$. We may then suppose that L is chosen large enough such that $x_{3,n} \geq a_n\left(1-\frac{L}{4}\delta_n\right)$ and then

$$|x - x_{3,n}| \leq a_n\left(1+\frac{L}{2}\delta_n\right) - a_n\left(1-\frac{L}{4}\delta_n\right)$$

$$\sim a_n\delta_n \sim \frac{a_n}{n}\Psi_n(x)$$

using (1.15) and (1.16).

Thus also in this case, it is possible to choose M such that (3.1) holds. Parts (ii) and (iii) of the lemma then follow similarly.□

Now fix M and δ in Lemma 3.1 and put

$$J_n := [x_{n,n}, x_{1,n}] \setminus \left[x_{k(x)+2},\ x_{k(x)-2}\right] \tag{3.7}$$

if $|x| \in \left[0, a_n\left(1+\frac{L}{2}\delta_n\right)\right]$ and

$$J_n := [x_{n,n}, x_{1,n}] \tag{3.8}$$

if $|x| \in [a_n(1+\frac{L}{2}\delta_n, \infty)$.

We modify the definition in (3.7) accordingly if $|x| \geq x_{1,n}$.

We have the following estimate.

Lemma 3.2. Uniformly for $1 \leq j \leq n$ and $n \geq N_0$,

$$\sum_{\substack{j=1 \\ j\notin[k(x)+2,k(x)-2]}}^{n} \frac{\Delta x_{j,n}}{|x-x_{j,n}|^{\alpha}} = \left\{\begin{array}{ll} O\left(a_n^{1-\alpha}\right) & ,0<\alpha<1 \\ O(\log n) & ,\alpha=1 \\ O\left(\frac{n}{a_n\Psi_n(x)}\right)^{\alpha-1} & ,\alpha>1. \end{array}\right\} \tag{3.9}$$

Proof. First note that if $|x| \leq a_n\left(1+\frac{L}{2}\delta_n\right)$,we have uniformly for $n \geq N_0$ and $1 \leq j \leq n$,

$$|x-t| \sim |x-x_{j,n}|\,, t \in [x_{j+1,n}, x_{j,n}]\,, j \notin [k(x)+2, k(x)-2]\,. \tag{3.10}$$

This follows much as in [3] using Lemma 3.1 (a) and (2.2) since,

$$\begin{aligned}\left|\frac{x-t}{x-x_{j,n}}-1\right| &= \left|\frac{t-x_{j,n}}{x-x_{j,n}}\right| \\ &\leq \left|\frac{x_{j,n}-x_{j+1,n}}{x-x_{j,n}}\right| \leq C\end{aligned}$$

and similarly we can bound

$$\frac{x-x_{j,n}}{x-t}.$$

Then, from (2.2) and the definition of J_n in (3.7), we obtain

$$\sum_{\substack{j=1 \\ j\notin[k(x)+2,k(x)-2]}}^{n} \frac{\Delta x_{j,n}}{|x-x_{j,n}|^{\alpha}} = O\left(\int_{\substack{|t|\leq a_n(1+A\delta_n) \\ t\in J_n}} \frac{dt}{|x-t|^{\alpha}}\right)$$

$$= \left\{\begin{array}{ll} O\left(a_n^{1-\alpha}\right) & ,0<\alpha<1 \\ O\left(\log n\right) & ,\alpha=1 \\ O\left(\frac{n}{a_n\Psi_n(x)}\right)^{\alpha-1} & \alpha>1. \end{array}\right\}.$$

The case for $|x|\geq a_n\left(1+\frac{L}{2}\delta_n\right)$ is similar but easier.□

We may now proceed with the proofs of our upper bounds. We begin with:

The Proof of the Upper bound in (1.17).

From (2.30) we have for $1\leq j\leq n$,

$$W(x)\, l_{j,n}(U_n)(x)\, W^{-1}(x_{j,n}) \leq C_1\left(\frac{\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n}{\left|1-\frac{|x|}{a_n}\right|+L\delta_n}\right)^{\frac{1}{4}} \frac{\Delta x_{j,n}}{|x-x_{j,n}|}.$$

Thus, by (1.6) and using the above, we have

$$\begin{aligned}\Lambda_n(W,U_n)(x) &= \sum_{j=1}^{n} W(x)\,|l_{j,n}(U_n)(x)|\, W^{-1}(x_{j,n}) \\ &\leq \sum_{j\in[k(x)+2,k(x)-2]} W(x)\,|l_{j,n}(U_n)(x)|\, W^{-1}(x_{j,n}) \\ &\quad + C_1 \sum_{j\notin[k(x)+2,k(x)-2]} \left(\frac{\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n}{\left|1-\frac{|x|}{a_n}\right|+L\delta_n}\right)^{\frac{1}{4}} \frac{\Delta x_{j,n}}{|x-x_{j,n}|}. \qquad (3.11)\end{aligned}$$

First observe that we may write

$$\frac{\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n}{\left|1-\frac{|x|}{a_n}\right|+L\delta_n}\leq 1+\frac{|x-x_{j,n}|}{a_n\left(\left|1-\frac{|x|}{a_n}\right|+L\delta_n\right)}. \tag{3.12}$$

Next we observe that using (2.10) , we may assume without loss of generality that $|x|\leq a_n$. Then (3.12) becomes using the definition of δ_n, [see (1.15)] ,

$$\begin{aligned}&\left(\frac{\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n}{\left|1-\frac{|x|}{a_n}\right|+L\delta_n}\right)^{\frac{1}{4}}\\ &\quad=O(1)+O\left(\frac{n^{\frac{1}{6}}T(a_n)^{\frac{1}{6}}|x-x_{j,n}|^{\frac{1}{4}}}{a_n^{\frac{1}{4}}}\right).\end{aligned} \tag{3.13}$$

Thus using (2.16) , (3.9) and (3.13) , we now rewrite (3.11) as,

$$\begin{aligned}\Lambda_n(W,U_n)(x) &\leq C_2\sum_{j\in[k(x)+2,k(x)-2]}1+O\left(\sum_{j\notin[k(x)+2,k(x)-2]}\frac{n^{\frac{1}{6}}T(a_n)^{\frac{1}{6}}\Delta x_{j,n}}{a_n^{\frac{1}{4}}|x-x_{j,n}|^{\frac{3}{4}}}\right)\\ &\quad+O\left(\sum_{j\notin[k(x)+2,k(x)-2]}\frac{\Delta x_{j,n}}{|x-x_{j,n}|}\right)\\ &= O(1)+O(\log n)+O\left(n^{\frac{1}{6}}T(a_n)^{\frac{1}{6}}\right)\\ &= O\left(n^{\frac{1}{6}}T(a_n)^{\frac{1}{6}}\right)\end{aligned} \tag{3.14}$$

and so we have taking sups,

$$\|\Lambda_n(W,U_n)\|_{L_\infty(\mathbb{R})}=O\left(n^{\frac{1}{6}}T(a_n)^{\frac{1}{6}}\right) \tag{3.15}$$

as required.□

We now present,

The Proof of our Upper bound in (1.24).

Firstly, from (2.31) we have for $1\leq j\leq n$,

$$\begin{aligned}&W(x)\,l_{j,n+2}(V_{n+2})(x)\,W^{-1}(x_{j,n})\\ &\quad\leq C_1\left(\frac{\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n}{\left|1-\frac{|x|}{a_n}\right|+L\delta_n}\right)^{\frac{-3}{4}}\frac{\Delta x_{j,n}}{|x-x_{j,n}|}.\end{aligned} \tag{3.16}$$

Thus by (1.6), (2.19), (2.20) and (3.16), we have

$$\Lambda_{n+2}\left(W, V_{n+2}\right)(x)$$
$$\leq O(1) + \sum_{\substack{j=1 \\ j\in[k(x)+2,k(x)-2]}}^{n} W(x)\left|l_{j,n+2}(V_{n+2})(x)\right| W^{-1}(x_{j,n})$$
$$+ C_2 \sum_{\substack{j=1 \\ j\notin[k(x)+2,k(x)-2]}}^{n} \left(\frac{\left|1-\frac{|x_{j,n}|}{a_n}\right| + L\delta_n}{\left|1-\frac{|x|}{a_n}\right| + L\delta_n}\right)^{\frac{-3}{4}} \frac{\Delta x_{j,n}}{|x-x_{j,n}|} \tag{3.17}$$
$$= O(1) + \sum_1 (x) + \sum_2 (x) \tag{3.18}$$

where

$$\sum_1 (x) := \sum_{\substack{j=1 \\ j\in[k(x)+2,k(x)-2]}}^{n} W(x)\left|l_{j,n+2}(V_{n+2})(x)\right| W^{-1}(x_{j,n})$$

and

$$\sum_2 (x) := C_2 \sum_{\substack{j=1 \\ j\notin[k(x)+2,k(x)-2]}}^{n} \left(\frac{\left|1-\frac{|x_{j,n}|}{a_n}\right| + L\delta_n}{\left|1-\frac{|x|}{a_n}\right| + L\delta_n}\right)^{\frac{-3}{4}} \frac{\Delta x_{j,n}}{|x-x_{j,n}|}.$$

We observe that using (2.11), we may assume without loss of generality that $|x| \leq a_{n+1}$. We begin with the estimation of $\sum_1 (x)$.

Note, that by (2.24), (2.33) and (2.34),

$$\begin{aligned} \sum_1 (x) &= \sum_{\substack{j=1 \\ j\in[k(x)+2,k(x)-2]}}^{n} \left|\frac{y_0^2 - x^2}{y_0^2 - x_{j,n}^2}\right| W(x)\left|l_{j,n}(U_n)(x)\right| W^{-1}(x_{j,n}) \\ &= O\Bigg(\sum_{\substack{j=1 \\ j\in[k(x)+2,k(x)-2]}}^{n} \left(\frac{\left|1-\frac{|x|}{a_n}\right| + L\delta_n}{\left|1-\frac{|x_{j,n}|}{a_n}\right| + L\delta_n}\right) \\ &\qquad \times W(x)\left|l_{j,n}(U_n)(x)\right| W^{-1}(x_{j,n}).\Bigg) \end{aligned} \tag{3.19}$$

Next, using (2.1), (2.2) and (2.4), it is easy to see that if L is large enough, we have uniformly for x and $j \in [k(x)+2, k(x)-2]$,

$$\left(\frac{\left|1-\frac{|x|}{a_n}\right|+L\delta_n}{\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n}\right) \sim 1$$

so that

$$\begin{aligned}\sum_1(x) &= O\left(\sum_{\substack{j=1\\ j\in[k(x)+2,k(x)-2]}}^{n} W(x)\left|l_{j,n}(U_n)(x)\right| W^{-1}(x_{j,n})\right)\\ &= O(1) \qquad (3.20)\end{aligned}$$

by (2.16).

We now turn to the delicate estimation of $\sum_2(x)$.

Much as in (3.12),we observe that for $1 \leq j \leq n$ we have

$$\left(\frac{\left|1-\frac{|x|}{a_n}\right|+L\delta_n}{\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n}\right)^{\frac{3}{4}} \leq 1+\frac{|x-x_{j,n}|^{\frac{3}{4}}}{a_n^{\frac{3}{4}}\left(\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n\right)^{\frac{3}{4}}}. \qquad (3.21)$$

Then, using (3.20), we may write

$$\sum_2(x) = O\left(\sum_{j\in S}\frac{\Delta x_{j,n}}{|x-x_{j,n}|}+\sum_{j\in S}\frac{\Delta x_{j,n}}{a_n^{\frac{3}{4}}|x-x_{j,n}|^{\frac{1}{4}}\left(\left|1-\frac{|x_{j,n}|}{a_n}\right|+L\delta_n\right)^{\frac{3}{4}}}\right)$$

where,

$$\begin{aligned}S &= \{j : 1 \leq j \leq n,\ j \notin [k(x)+2, k(x)-2]\},\\ &= O(\log n)+O\left(\sum_{j\in S}\frac{\Delta x_{j,n}}{|x-x_{j,n}|^{\frac{1}{4}}\left(|a_n-|x_{j,n}||+a_nL\delta_n\right)^{\frac{3}{4}}}\right)\\ &\qquad \text{by } (3.9)\\ &= O(\log n)+O\left(\sum_{\substack{j\in S\\ |x_{j,n}|\leq a_n(1-\delta_n)}}\frac{\Delta x_{j,n}}{|x-x_{j,n}|^{\frac{1}{4}}\left(a_n-|x_{j,n}|\right)^{\frac{3}{4}}}\right)\end{aligned}$$

$$+O\left(\sum_{\substack{j\in S\\ |x_{j,n}|>a_n(1-\delta_n)}}\frac{\Delta x_{j,n}n^{\frac{1}{2}}T\left(a_n\right)^{\frac{1}{2}}}{|x-x_{j,n}|^{\frac{1}{4}}\,a_n^{\frac{3}{4}}}\right). \tag{3.22}$$

Next, using the Geometric and Arithmetic mean inequality and (3.9) again, we may continue (3.21) as

$$\begin{aligned}\sum_2(x) &= O\left(\log n\right)\\ &\quad+O\left(\sum_{\substack{j\in S\\ |x_{j,n}|\le a_n(1-\delta_n)}}\frac{\Delta x_{j,n}}{|x-x_{j,n}|}\right)+O\left(\sum_{\substack{j\in S\\ |x_{j,n}|\le a_n(1-\delta_n)}}\frac{\Delta x_{j,n}}{a_n-|x_{j,n}|}\right)\\ &\quad+O\left(\frac{n^{\frac{1}{2}}T\left(a_n\right)^{\frac{1}{2}}}{a_n^{\frac{3}{4}}}\sum_{\substack{j\in S\\ |x_{j,n}|>a_n(1-\delta_n)}}\frac{\Delta x_{j,n}}{|x-x_{j,n}|^{\frac{1}{4}}}\right)\\ &= O\left(\log n\right)+O\left(\sum_{\substack{j\in S\\ |x_{j,n}|>a_n(1-\delta_n)}}1\right) \end{aligned}\tag{3.23}$$

where in the last line we used $(1.15), (1.16), (2.1)$ and (2.2).

Now it remains to observe that the spacing (2.2) and (1.16), imply that there exist at most a finite number of j such that $|x_{j,n}|>a_n\left(1-\delta_n\right)$. Then (3.22) yields,

$$\sum_2(x)=O\left(\log n\right)+O\left(1\right)=O\left(\log n\right). \tag{3.24}$$

Combining (3.23) with (3.19) and taking sups yields

$$\left\|\Lambda_{n+2}\left(W,V_{n+2}\right)\right\|_{L_\infty(\mathbb{R})}=O\left(\log n\right) \tag{3.25}$$

as required.□

4 The Proofs of Theorems 1.2 and 1.4 and Corollaries 1.3 and 1.5.

In this section we present the proofs of our lower bounds in (1.17) and (1.24). We deduce Theorems 1.2 and 1.4 and Corollaries 1.3 and 1.5.

We begin with,

The Proof of our lower bound in (1.17).
Write

$$\Lambda_n(W,U_n)(x) = W(x)|p_n(x)|\sum_{j=1}^{n} p_n'(x_{j,n})^{-1} W(x_{j,n})^{-1}|x-x_{j,n}|^{-1}. \quad (4.1)$$

In particular, (4.1) becomes using $(1.16), (2.9), (2.12)$ and (2.17),

$$\begin{aligned}\Lambda_n(W,U_n)(y_0) &\geq C_1 \sum_{0\leq x_{j,n}\leq \frac{a_n}{2}} \frac{a_n^{\frac{-1}{2}} n^{\frac{1}{6}} T(a_n)^{\frac{1}{6}}}{a_n^{\frac{-1}{2}}\left(1-\frac{|x_{j,n}|}{a_n}+L\delta_n\right)^{\frac{-1}{4}} n} \\ &\geq C_2 n^{\frac{-5}{6}} T(a_n)^{\frac{1}{6}} \sum_{0\leq x_{j,n}\leq \frac{a_n}{2}} 1. \quad (4.2)\end{aligned}$$

Now it remains to observe that the spacing (2.2) and (1.16) imply that there exist $\geq C_3 n$ j such that $x_{j,n}\in[0,\frac{a_n}{2}]$. Then (4.2) becomes

$$\Lambda_n(W,U_n)(y_0)\geq C_4 n^{\frac{1}{6}} T(a_n)^{\frac{1}{6}}$$

so that

$$\|\Lambda_n(W,U_n)\|_{L_\infty(\mathbb{R})}\geq \Lambda_n(W,U_n)(y_0)\geq C_5 n^{\frac{1}{6}} T(a_n)^{\frac{1}{6}}, \quad (4.3)$$

as required.□

We now turn to the proof of our lower bound (1.24). Here a choice of $x=y_0$ is not sufficient to achieve our lower bound and we need to proceed more carefully. Indeed, we will show that the point we need sits "far" away from a_n.

The Proof of our lower bound for (1.24).
First we claim that there exists $y\in\mathbb{R}$ satisfying $|y|\leq \alpha a_n$, for some $0<\alpha<1$ and uniformly for $n\geq 1$,

$$a_n^{\frac{1}{2}} p_n W(y)\sim 1. \quad (4.4)$$

To see this, observe first that if $0 < \alpha < 1$ is given, then by $(1.16), (2.2)$ and (2.9), there exists $> C_1 n\ \ j$, $1 \leq j \leq n+1$ such that $|x_{j,n+1}| \in [0, \alpha a_n]$. Now choose $y = y_1 = x_{k,n+1}$ for some $1 \leq k \leq n+1$ such that $|y_1| \in [0, \alpha a_n]$. Then (2.9) and (2.17) give

$$a_n^{\frac{1}{2}} |p_n W| (y_1) \sim 1$$

and (4.4) is established. Fix y_1 as above.

We now proceed as follows. Since $y_1 < c y_0$, for some $0 < c < 1$, we have by $(1.29), (2.12), (2.33)$ and (4.4),

$$\begin{aligned}
\Lambda_{n+2}(W, V_{n+2})(y_1) &\geq \sum_{j=1}^{n} W(y_1) W(x_{j,n})^{-1} \left(\frac{y_0^2 - y_1^2}{y_0^2} \right) l_{j,n}(U_n)(y_1) \\
&\geq C_1 \sum_{j=1}^{n} W(y_1) W(x_{j,n})^{-1} l_{j,n}(U_n)(y_1) \\
&\geq C_2 \Lambda_n(W, U_n)(y_1) \geq C_3 a_n^{\frac{1}{2}} |p_n W| (y_1) \log n \\
&\geq C_4 \log n.
\end{aligned}$$

Thus,

$$\|\Lambda_{n+2}(W, V_{n+2})\|_{L_\infty(\mathbb{R})} \geq C_4 \log n \tag{4.5}$$

and we have proved our lower bound.□

We may now present:

The Proof of Theorem 1.2.

This follows immediately from (3.15) and (4.3) □

The Proof of Corollary 1.3.

(1.18) follows from the representation $(1.6), (1.17)$ and Theorem 1.2 of [4]. (1.21) and (1.22) follow from (1.18) ,Corollary 1.7 of [3] and (2.7) .□

The Proof of Theorem 1.4.

This follows immediately from (3.24) and (4.5) .□

The Proof of Corollary 1.5.

(1.25) follows from the representation $(1.6), (1.24)$ and Theorem 1.2 of [4]. (1.26) and (1.27) follow from (1.25) ,Corollary 1.7 of [3] and (2.7) .□

5 Pointwise estimates of $\Lambda_n(W, U_n)$

In this section, we sketch briefly the proof of Theorem 1.6.

Fix $x, x_{k(x),n}, M, \delta$ and J_n as in Section 3.

Step 1: Set

$$S_1 := \left\{ j : 1 \leq j \leq n, |x - x_{j,n}| \leq \frac{\delta a_n}{n} \Psi_n(x) \right\},$$

$$S_2 := \left\{ j : 1 \leq j \leq n, \frac{\delta a_n}{n} \Psi_n(x) \leq |x - x_{j,n}| \leq \frac{M a_n}{n} \Psi_n(x) \right\}$$

and

$$S_3 := \left\{ j : 1 \leq j \leq n, |x - x_{j,n}| > \frac{M a_n}{n} \Psi_n(x) \right\}.$$

Now write:

$$\Lambda_n(U_n, W)(x) := \sum_{j \in S_1}(x) + \sum_{j \in S_2}(x) + \sum_{j \in S_3}(x).$$

Step 2: Estimation of $\sum_{j \in S_1}(x)$ and $\sum_{j \in S_2}(x)$.

First observe that it suffices to estimate the above sums for $x \in \left[0, a_n \left(1 + \frac{L}{2}\delta_n\right)\right]$ for they are identically zero outside this range of x. Moreover, recall that we may assume by symmetry that $x > 0$.

Then the following holds:

Lemma 5.1. Let $W \in \mathcal{E}$.

(a) There exists $C_1 \geq 0$ such that uniformly for $n \geq 1$ and $x \in \left[0, a_n \left(1 + \frac{L}{2}\delta_n\right)\right]$,

$$0 \leq \sum_{j \in S_1}(x) \leq C_1. \tag{5.1}$$

Moreover, uniformly for $n \geq 1$ and $x \in [0, x_{1,n}]$,

$$\sum_{j \in S_1}(x) \sim 1. \tag{5.2}$$

(b) Uniformly for $x \in \left[0, a_n \left(1 + \frac{L}{2}\delta_n\right)\right]$ and $n \geq N_0$,

$$\sum_{j \in S_2} (x) \sim \sqrt{a_n} \left| p_n W \right| (x) \left(1 - \frac{|x|}{a_n} + L\delta_n\right)^{\frac{1}{4}}. \tag{5.3}$$

Proof. First note that (2.16) gives

$$\begin{aligned} \sum_{j \in S_1} (x) &= W(x) \sum_{j \in S_1} \left| l_{j,n} (U_n)(x) \right| W^{-1}(x_{j,n}) \\ &\leq C \sum_{j \in S_1} 1 \leq C_1 \end{aligned}$$

for some $C_1 > 0$ independent of x and n as the above sum is finite.

For the lower sum, we use the weighted Erdős-Turan inequality (see for example [5]),

$$l_{j,n}(U_n)(x) W(x) W^{-1}(x_{j,n}) + l_{j+1,n}(U_n)(x) W(x) W^{-1}(x_{j+1,n}) \geq 1 \tag{5.4}$$

valid for $n \geq 2, 1 \leq j \leq n-1$ and $x \in [x_{j+1,n}, x_{j,n}]$.

If $x \leq x_{1,n}$,we may assume without loss of generality that $x \in \left[x_{k(x)+1,n}, x_{k(x),n}\right]$. Then (5.4) gives

$$\sum_{j \in S_1} (x) \geq W(x) \sum_{j=k(x)}^{k(x)+1} l_{j,n}(U_n)(x) W^{-1}(x_{j,n}) \geq C_2.$$

Thus (5.1) and (5.2) follow.

It remains to show (5.3). Here we first observe that by (2.2) we have uniformly for $j \in S_2$,

$$\frac{a_n}{n} \Psi_n(x_{j,n}) \sim |x_{j,n} - x_{j\pm 1,n}| \sim |x - x_{j,n}|. \tag{5.5}$$

Then (2.15) and (5.5) easily yield

$$\sum_{j \in S_2} (x) \sim \sqrt{a_n} \left| p_n W \right| (x) \left(1 - \frac{|x|}{a_n} + L\delta_n\right)^{\frac{1}{4}}$$

as required.□

Preliminary estimation of $\sum_{j\in S_3}(x)$.

Lemma 5.2. Let $W \in \mathcal{E}$.

(a) If $|x| \leq 2a_n$,we have uniformly for x and $n \geq N_0$,

$$\sum_{j\in S_3}(x) \sim \sqrt{a_n}p_n W(x) \int_{\substack{|t|\leq a_n(1+A\delta_n)\\ t\in J_n}} \frac{\left(1-\frac{|t|}{a_n}+L\delta_n\right)^{\frac{1}{4}}}{|x-t|}dt. \tag{5.6}$$

(b) If $|x| \leq 2a_n$,we have uniformly for x and $n \geq N_0$,

$$\sum_{j\in S_3}(x) \sim \frac{\sqrt{a_n}p_n W(x)}{|x|} \int_{\substack{|t|\leq a_n(1+A\delta_n)\\ t\in J_n}} \left(1-\frac{|t|}{a_n}+L\delta_n\right)^{\frac{1}{4}} dt. \tag{5.7}$$

Proof. We consider the case $x \in \left[0, a_n\left(1+\frac{L}{2}\delta_n\right)\right]$ and $x \leq x_{1,n}$. The other cases are similar.

By (2.2) and (2.15),

$$\sum_{j\in S_3}(x) \sim \sqrt{a_n}p_n W(x) \sum_{j\in[1,n]\backslash[k(x)+2,k(x)-2]} \int_{x_{j+1,n}}^{x_{j,n}} \frac{\left(1-\frac{|x_{j,n}|}{a_n}+L\delta_n\right)^{\frac{1}{4}}}{|x-x_{j,n}|}dt. \tag{5.8}$$

Then much as in (3.10), (5.8) readily yields (5.6) for this case.□

Step 4: Estimation of

$$J := \int_{\substack{|t|\leq a_n(1+A\delta_n)\\ t\in J_n}} \left(1-\frac{|t|}{a_n}+L\delta_n\right)^{\frac{1}{4}} dt.$$

We now record the following technical estimate for J :

Lemma 5.3. Let $W \in \mathcal{E}$ and suppose that $x \in \left[0, a_n\left(1+\frac{L}{2}\delta_n\right)\right]$. Then uniformly for x and $n \geq N_0$,

$$J \sim \left(1-\frac{|x|}{a_n}+L\delta_n\right)^{\frac{1}{4}} \log\left(\frac{n\left(1-\frac{|x|}{a_n}+L\delta_n\right)}{\Psi_n(x)}\right)+1. \tag{5.9}$$

Step 5: The Proof of Theorem 1.6.

Observe that for $|x| \leq a_n \left(1 + \frac{L}{2}\delta_n\right)$,

$$\log\left(\frac{n\left(1 - \frac{|x|}{a_n} + L\delta_n\right)}{\Psi_n(x)}\right) > 0 \text{ if } L \text{ is large enough.}$$

Then $(5.1), (5.2), (5.3)$ and (5.9) yield the result for this case. Theorem 1.6 (b) and (c) are similar but easier.□

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