Marchaud Inequalities for a class of Erdős Weights.

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Abstract

We prove Marchaud Inequalities for a class of Erdős Weights. The main feature of these weights is that they are of faster than polynomial growth at infinity. We also survey some other classical properties of our modulus of continuity.

1 Introduction and Statement of Results

An Erdős Weight is of the form: $W(x) := \exp\left[-Q(x)\right]$, where $Q : \mathbb{R} \longrightarrow \mathbb{R}$ is even, $xQ'(x)$ is increasing and

$$Q > x^r,\ r > 0,\ x \geq x_0.$$

That is, Q is of faster than polynomial growth at infinity [5,6,7]. In particular,

$$W_{k,\alpha}(x) := \exp\left(-\exp_k\left(-|x|^\alpha\right)\right) \quad \alpha > 1 \tag{1}$$

and

$$W_{A,B}(x) := \exp\left(-\exp\left[\log\left(A + x^2\right)\right]^B\right) \quad B > 1,\ A \text{ large enough} \tag{2}$$

are examples of Erdős Weights. Here, $\exp_k = \exp\left(\exp\left(\ldots\exp\left(\right)\right)\right)$ denotes the kth iterated exponential. In [1,2], we considered the subject of weighted polynomial approximation for Erdős Weights, W, and estimated the quantity,

$$E_n[f]_{W,p} := \inf_{P \in \Pi_n} \left\|(f - P)\,W\right\|_{L_p(\mathbb{R})} \tag{3}$$

where, $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a suitable real valued function, Π_n denotes the class of polynomials of degree $\leq n$, $n \geq 1$ and $0 < p \leq \infty$. In this paper, we continue this investigation by presenting some interesting properties of our weighted modulus of continuity.

We first however, need some notation and backround.

1.1 Notation and Backround

1.1.1 a_n

Given W an Erdős Weight, set for $u > 0$, a_u to be the Mhaskar–Rakhmanov–Saff number defined as the positive root of,

$$u = \frac{2}{\pi} \int_0^1 \frac{a_u t Q'(a_u t)}{\sqrt{1-t^2}} dt. \tag{4}$$

For example, for $W(x) = W_{k,\alpha}(x)$, $a_n = a_n(Q_{k,\alpha})$ satisfies,

$$a_n = \left[\log_{k-1}\left(\log n - \frac{1}{2}\sum_{j=2}^{k+1} \log_j n \quad + O(1)\right)\right]^{\frac{1}{\alpha}} \tag{5}$$

where $\log_j = \log(\log ... (\log()))$ denotes the jth iterated logarithm.The importance of a_n lies in the fact that for $P \in \Pi_n$, $n \geq 1$, the quantity (PW) "lives" mostly on $[-a_n, a_n]$ [8].

1.1.2 Moduli and K-Functionals

To prove Jackson and Converse theorems for Erdős Weights, we need a suitable weighted modulus of continuity. This was first introduced in [4], in a similar context and suitably modified in [1, 3]. For $h > 0$ and $r \geq 1$, let

$$\Delta_h^r f(x) := f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

and

$$\Delta_h^r f(x) := \sum_{j=0}^{r} \binom{r}{j} (-1)^j f\left(x + \left(\frac{r}{2} - j\right) h\right)$$

denote respectively, the first and rth order symmetric differences of a real valued $f : \mathbb{R} \longrightarrow \mathbb{R}$. Further, set:

(1)

$$T(x) := \frac{xQ'(x)}{Q(x)} \quad x \in \mathbb{R} \tag{6}$$

satisfying $T(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

(2)

$$\sigma(t) := \inf\left\{a_u : \frac{a_u}{u} \leq t\right\} \quad t > 0. \tag{7}$$

(3)

$$\Psi_n(x) := \left|1 - \left(\frac{|x|}{a_n}\right)^2\right| + T(a_n)^{-1}, \quad x \in \mathbb{R}, \quad n \geq 1. \tag{8}$$

(4)

$$\Phi_t(x) := \left|1 - \frac{|x|}{\sigma(t)}\right|^{\frac{1}{2}} + T(\sigma(t))^{-\frac{1}{2}}\ ,\ x \in \mathbb{R},\quad t > 0. \tag{9}$$

Then we define for $0 < p \le \infty,\ t > 0$ and $r \ge 1$ our weighted modulus of continuity,

$$\begin{aligned} w_{r,p}(f, W, t) \quad : \quad &= \sup_{0<h\le t} \left\| W\Delta_{h\Phi_t(x)} f \right\|_{L_p(|x|\le\sigma(t)+rt)} \\ &+ \inf_{P\in\Pi_{r-1}} \|(f-P)W\|_{L_p(|x|\ge\sigma(t))} \end{aligned} \tag{10}$$

and corresponding weighted realisation-functional for W given by:

$$K_{r,p}(f, W, t^r) := \inf_{P\in\Pi_n} \left\{ \|(f-P)W\|_{L_p(\mathbb{R})} + t^r \left\| P^{(r)} \Psi_n^{\frac{r}{2}} W \right\|_{L_p(\mathbb{R})} \right\} \tag{11}$$

where,

$$n := n(t) = \inf\left\{k :\ \frac{a_k}{k} \le t\right\} \tag{12}$$

$[1, 2, 3, 4, 6]$.

We remark that it is shown in $[1, 2]$, that if $t > 0$ is small enough and n is determined by (12) then,

$$\frac{a_n}{n} \le t < 2\frac{a_n}{n}. \tag{13}$$

1.1.3 Jackson Theorem, and Realization-functional equivalence

For a wide class of Erdős Weights, $\mathcal{E}$, which satisfy [in particular],

(a)

$$W_{k,\alpha},\ W_{A,B}\ \in \mathcal{E},$$

[recall (1) and (2)],

(b)

$$\begin{aligned} &a_u u^{-\varepsilon} \longrightarrow 0,\quad u \longrightarrow \infty\ \ \forall\varepsilon > 0 \\ &\left(\frac{a_{\frac{u}{2}}}{a_u}\right) \longrightarrow 1,\quad u \longrightarrow \infty, \end{aligned} \tag{14}$$

(c)

$$\sup_{x\in\mathbb{R}} \frac{\Psi_n(x)}{\Psi_m(x)} \le C \log\left(2 + \frac{n}{m}\right) \tag{15}$$

for $m \le n$, and the Markov-Bernstein inequality

(d)

$$\left\| P^{(r+1)} W \Phi_n^{\frac{r}{2}} \right\|_{L_p(\mathbb{R})} \leq C_1 \frac{n}{a_n} \left\| P^{(r)} W \right\|_{L_p(\mathbb{R})} \tag{16}$$

$0 < p \leq \infty,\ r \geq 1,\ P \in \Pi_n$ and some $C_1 > 0\,[2,5,7]$, we note the following unpublished results of $[1,2]$.

__**Theorem 1.1**__[Jackson Theorem]

Let $W \in \mathcal{E},\ n \geq N$ be given by (12), $r \geq 1$ and $0 < p \leq \infty$. Then $\exists\ C_2 > 0,\ C_2 \neq C_2(f,t)$ such that,

$$E_n[f]_{W,p} \leq C_2 w_{r,p}\left(f, W, \frac{a_n}{n}\right). \tag{17}$$

__**Theorem 1.2**__[Equivalence of K-functional and modulus]

Let $W \in \mathcal{E},\ L > 0,\ r \geq 1$ and $0 < p \leq \infty$. Then $\exists\ C_3, C_4, C_5 > 0$ independent of f and t, such that for small enough $t > 0$,

$$w_{r,p}(f, W, Lt) \leq C_3 K_{r,p}(f, W, t^r) \leq C_4 w_{r,p}(f, W, C_5 t). \tag{18}$$

Naturally, it is (18) that allows us to deduce:

__**Corrolary 1.3.**__

Let $W \in \mathcal{E},\ r \geq 1,\ 0 < p \leq \infty$. Let $t > 0$ be small enough and n determined by (12). Then there exists $C_5, C_6 > 0$ independent of t and n such that uniformly in f,

$$C_5 \leq \frac{w_{r,p}\left(f, W, \frac{a_n}{n}\right)}{w_{r,p}(f, W, t)} \leq C_6. \tag{19}$$

2 Statement of Results.

In this paper we use (18) and (19) to deduce a *Marchaud Inequality* for our modulus. Here is our theorem:

__**Theorem 1.4.**__ [Marchaud Inequality]

Let $W \in \mathcal{E}$ and further suppose that W satisfies (16) and (17). Let $q = \min(1,p)$, $0 < p \leq \infty$. Then for some C_7, C_8 independent of f and t and t small enough,

$$w_{r,p}(f, W, t) \leq C_7 t^r \left[\int_t^{C_8} \frac{w_{r+1,p}(f, W, u)^q}{u^{rq} \left(\log_2\left(\frac{1}{ut}\right)\right)^{\frac{rq}{2}}} du + \left(\log_2\left(\frac{1}{tr}\right)\right)^{\frac{rq}{2}} \|fW\|_{L_p(\mathbb{R})}^q \right]^{\frac{1}{q}} \tag{20}$$

for all $f : \mathbb{R} \longrightarrow \mathbb{R}$ for which (10) and (11) are meaningful.

3 Proof of Theorem 1.4

First let n be large enough and let P_n^* be the best approximant to f which exists and satisfies,

$$E_n[f]_{W,p} := \|(f-P)W\|_{L_p(\mathbb{R})}. \tag{21}$$

By $(11), (17)$ and (18) we may thus write using (21),

$$\begin{aligned} & w_{r,p}\left(f, W, \frac{a_n}{n}\right)^q && (22) \\ \leq\ & C_9\left[\|(f-P_n^*)W\|_{L_p(\mathbb{R})} + \left(\frac{a_n}{n}\right)^{rq}\left\|P_n^{*(r)}\Psi_n^{\frac{r}{2}}W\right\|_{L_p(\mathbb{R})}\right] \\ \leq\ & C_{10}w_{r+1,p}\left(f, W, \frac{a_n}{n}\right)^q + \left(\frac{a_n}{n}\right)^{rq}\left\|P_n^{*(r)}\Psi_n^{\frac{r}{2}}W\right\|_{L_p(\mathbb{R})} \end{aligned}$$

for some $C_9, C_{10} > 0$.Here we use the inequality $(a+b)^\alpha \leq a^\alpha + b^\alpha \quad a, b > 0,\ 0 < \alpha < 1$. Now choose $l = l(n)$ such that,

$$r2^{l+2} \geq n \geq r2^{l+1} \tag{23}$$

where $n \geq 2r$, and write,

$$P_n^*(x) = \sum_{k=0}^{l-1}\left(P^*_{\left[\frac{n}{2^k}\right]}(x) - P_{\left[\frac{n}{2^{k+1}}\right]}(x)\right) + P_{\left[\frac{n}{2^{l+1}}\right]}(x) \tag{24}$$

where $[x]$ =the largest integer $\leq x$.

Using (16) and (21) gives for $0 \leq k \leq l$,

$$\begin{aligned} & \left\|\left(P^*_{\left[\frac{n}{2^k}\right]}(x) - P^*_{\left[\frac{n}{2^{k+1}}\right]}(x)\right)W\right\|^q_{L_p(\mathbb{R})} && (25) \\ \leq\ & \left\|\left(f - P^*_{\left[\frac{n}{2^{k+1}}\right]}(x)\right)W\right\|^q_{L_p(\mathbb{R})} + \left\|\left(P^*_{\left[\frac{n}{2^k}\right]}(x) - f\right)W\right\|^q_{L_p(\mathbb{R})} \\ \leq\ & C_{11}w_{r+1,p}\left(f, W, \frac{a_{\left[\frac{n}{2^{k+1}}\right]}}{\left[\frac{n}{2^{k+1}}\right]}\right)^q \end{aligned}$$

for some $C_{11} > 0$. Keeping in mind (22), we may now combine $(15), (16), (24)$ and (25) to give,

$$\begin{aligned} \left\|P_n^{*(r)}\Psi_n^{\frac{r}{2}}W\right\|^q_{L_p[\mathbb{R}]} \leq\ & C_{12}\left[\left\|\sum_{k=0}^{l-1}\left(P^{*(r)}_{\left[\frac{n}{2^k}\right]}(x) - P^{*(r)}_{\left[\frac{n}{2^{k+1}}\right]}(x)\right)\Psi_n^{\frac{r}{2}}W\right\|^q_{L_p(\mathbb{R})}\right. && (26) \\ & \left. + \left\|P^{*(r)}_{\frac{n}{2^l}}(x)\Psi_n^{\frac{r}{2}}W\right\|^q_{L_p(\mathbb{R})}\right] \end{aligned}$$

$$
\begin{aligned}
&\le C_{13}\left[\left\|\sum_{k=0}^{l-1}(k+2)^{rq}\left(P^{*(r)}_{\left[\frac{n}{2^k}\right]}(x)-P^{*(r)}_{\left[\frac{n}{2^{k+1}}\right]}(x)\right)\Psi^{\frac{r}{2}}_{\frac{n}{2^k}}W\right\|^q_{L_p(\mathbb{R})}\right. &(27)\\
&\qquad\left.+\left\|l^{rq}P^{*(r)}_{\frac{n}{2^l}}(x)\,\Psi^{\frac{r}{2}}_{\frac{n}{2^l}}W\right\|^q_{L_p(\mathbb{R})}\right] &(28)\\
&\le C_{14}\left[\sum_{k=0}^{l-1}\left(\frac{a_{\left[\frac{n}{2^{k+1}}\right]}}{\left[\frac{n}{2^{k+1}}\right]}\right)^{rq}(k+2)^{rq}\left\|\left(P^{*}_{\left[\frac{n}{2^k}\right]}(x)-P^{*}_{\left[\frac{n}{2^{k+1}}\right]}(x)\right)W\right\|^q_{L_p(\mathbb{R})}\right.\\
&\qquad\left.+\left(\frac{a_{\left[\frac{n}{2^l}\right]}}{\left[\frac{n}{2^l}\right]}\right)^{rq}l^{rq}\|fW\|^q_{L_p(\mathbb{R})}\right]
\end{aligned}
$$

for some C_{12}, C_{13} and $C_{14}>0$.

We may now combine (26) with (22) and express this as an integral with the help of (14) and (23) as,

$$
w_{r,p}\left(f,W,\frac{a_n}{n}\right)^q \le C_{15}\left(\frac{a_n}{n}\right)^{rq}\left[\int_{\frac{a_n}{n}}^{C_{16}}\frac{w_{r+1,p}(f,W,u)^q}{u^{rq}\left(\log_2\left(\frac{n}{u}\right)\right)^{\frac{rq}{2}}}du\ +\left(\log_2\left(\frac{n}{r}\right)\right)^{\frac{rq}{2}}\|fW\|^q_{L_p(\mathbb{R})}\right]^{\frac{1}{q}} \tag{29}
$$

whereby following the proof carefully, it may be easily seen that C_{15} and C_{16} are independent of f and t. Now let $t>0$ be small enough and determine n by (12). First observe that using (13) and (14), we deduce that there exists constants C_{17} and $C_{18}>0$ independent of t and n such that,

$$
C_{18}\le\frac{\log n}{\log\left(\frac{1}{t}\right)}\le C_{17} \tag{30}
$$

so that using (13), (19) and (28), (27) becomes,

$$
w_{r,p}(f,W,t)^q\le C_{19}(t)^{rq}\left[\int_{\frac{a_n}{n}}^{C_{20}}\frac{w_{r+1,p}(f,W,u)^q}{u^{rq}\left(\log_2\left(\frac{1}{ut}\right)\right)^{rq}}du\ +\left(\log_2\left(\frac{1}{tr}\right)\right)^{rq}\|fW\|^q_{L_p(\mathbb{R})}\right]^{\frac{1}{q}}.
$$

Taking qth roots gives the result.□

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