Hilbert Transforms and Orthonormal Expansions for Exponential Weights

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Abstract. We establish the uniform boundedness of the weighted Hilbert transform for a general class of symmetric and nonsymmetric weights on a finite or infinite interval $I := (c, d)$ with $c < 0 < d$. We then apply these results to study mean and uniform convergence of orthonormal expansions on the line.

§1. Introduction and Statement of Results

In this paper we shall study mean and uniform convergence of orthonormal expansions as well as uniform bounds for the weighted Hilbert transform for a large class of exponential decaying weights on an interval $I := (c, d)$ with $c < 0 < d$. For orthonormal expansions on $[-1, 1]$, there are many well known results for Chebyshev, Jacobi and generalized Jacobi weights starting from Riesz, and we do not review these here. Instead, we refer the reader to [19,20,24,28,32] and the many references cited therein for a detailed and comprehensive account of this vast and interesting subject. Our study involves exponentially decaying weights w on I and functions f which may grow at $\pm\infty$ or near the endpoints of I. More precisely, in Theorems 1.2 and 1.7 below, we study both mean and uniform convergence of orthonormal expansions for a class of symmetric exponential weights on the line of polynomial decay at infinity, the so called Freud class which will be defined more precisely in Definition 1.1 below. Our mean convergence results are motivated by a recent result of Jha and Lubinsky in [20, Theorem 1.2] and indeed we will show how to improve this latter result. Our uniform convergence result, see Theorem 1.7, relies on good uniform bounds for the weighted Hilbert transform, see Theorems 1.6(A-B). We establish these latter bounds for a general class of symmetric and nonsymmetric weights on I.

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1.1 Statement of Results

To set the scene for our investigations, let $I := (c, d)$ where $c < 0 < d$ with c and d finite or infinite. (Note that I need not be symmetric about 0 but should contain 0). Let w be a nonnegative weight on I with $x^n w(x) \in L^1$, $n = 0, 1, ...$ The idea of this paper arose, partly, from an interesting paper of Geza Freud [18] who studied the dependence of the greatest zero of $p_n(w^2)$, the nth orthonormal polynomial for w^2, on the corresponding recurrence coefficients. More precisely, given w as above, we recall, see [19,30], that $p_n := p_n(w^2)$ admits the representation

$$p_n(x) = \gamma_n x^n + \cdots, \quad \gamma_n := \gamma_n(w^2) > 0$$

and satisfies

$$\int_I p_n(x) p_n(x) w^2(x) dx = \delta_{m,n}, \qquad m, n \geq 0.$$

We denote by $x_{j,n} := x_{j,n}(w^2)$, $1 \leq j \leq n$, the jth simple zero of p_n in I, and we order these zeroes as

$$x_{n,n} < x_{n-1,n} \cdots < x_{2,n} < x_{1,n}.$$

If w is even, we write the three term recurrence of p_n in the form

$$x p_n(x) = A_n(w^2) p_{n+1}(x) + A_{n-1}(w^2) p_{n-1}(x), \qquad n \geq 0.$$

Here, $p_{-1} = 0$, $p_0 = \left(\int w^2(x)\, dx\right)^{-1/2}$ and

$$A_n := A_n(w^2) = \gamma_{n-1}/\gamma_n > 0$$

are the corresponding recurrence coefficients. Given w as above, we may also form an orthonormal expansion

$$f(x) \to \sum_{j=0}^{\infty} b_j p_j(x),\ b_j := \int_I f p_j w,\ j \geq 0 \tag{1.1}$$

for any measurable function $f : I \to (-\infty, \infty)$ for which

$$\int_I |f(x) x^j| w(x) dx < \infty, \qquad j = 1, 2, 3... \tag{1.2}$$

Let $S_n[.w] := S_n[.], n \geq 1$ denote the nth partial sums of the orthonormal expansions given by (1.2).

To state our main results, we require some additional notation. To this end, let us agree that henceforth C will denote a positive constant independent of x, y, n, f and j which may take on different values at different times. Moreover, for any two sequences b_n and c_n of nonzero real numbers, we shall write $b_n = O(c_n)$ if there exists a positive constant C, independent of n, such that

$$b_n \leq Cc_n, \qquad n \to \infty$$

and $b_n \sim c_n$ if

$$b_n = O(c_n) \qquad \text{and} \qquad c_n = O(b_n).$$

Henceforth, for functions and sequences of functions, O and $\sim$ will be uniform in x, y, n, f and j.

We begin with the definition of a large class of *admissible* weights, see Definitions 1.1 (A-C) below. For clarity of exposition, explicit and easily absorbed examples of admissible weights are presented immediately after Definitions 1.1(A-C).

Definition 1.1A. *A function $g : (0, d) \to (0, \infty)$ is said to be* quasi-increasing *if*

$$g(x) \leq Cg(y), \qquad 0 < x \leq y < d.$$

It is easy to see that any increasing function is quasi-increasing. Similarly, we may define the notion of quasi-decreasing.

Definition 1.1B. *A weight function $w : I \to (0, \infty)$ will be called* admissible *if each of the following conditions below is satisfied:*

a) *$Q := \log(1/w)$ is continuously differentiable and satisfies $Q(0) = 0$;*

b) *Q' is nondecreasing in I with*

$$\lim_{x \to c^+} Q(x) = \lim_{x \to d^-} Q(x) = \infty;$$

c) *The function*

$$T(x) := \frac{xQ'(x)}{Q(x)},\ x \neq 0$$

is quasi-increasing in $(0, d)$ and quasi decreasing in $(c, 0)$ with

$$T(x) \geq \lambda > 1,\ x \in I \backslash 0;$$

d) *Given a sufficiently small $B > 0$ along with positive ε and δ,*

$$\frac{\log|Q'(x+\delta)|}{(|Q'(x-\varepsilon)|)^B} \leq C,$$

for all x close enough to c and d.

Definition 1.1B suffices for uniform bounds on the weighted Hilbert transform, see Theorems 1.6(A-B) below. For Theorems 1.2, 1.3, 1.5 and 1.7, we need some additional smoothness and regularity assumptions on Q'.

Definition 1.1C. *An admissible weight w will be called* strongly admissible *if the following additional assumptions on w hold:*

a) *There exists $\varepsilon_0 \in (0,1)$ such that for $y \in I\backslash\{0\}$*

$$T(y) \sim T\left(y\left[1-\frac{\varepsilon_0}{T(y)}\right]\right);$$

b) *For every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x \in I\backslash\{0\}$,*

$$\int_{x-\frac{\delta x}{T(x)}}^{x+\frac{\delta x}{T(x)}} \frac{|Q'(s)-Q'(x)|}{|s-x|^{3/2}} ds \leq \varepsilon |Q'(x)| \sqrt{\frac{T(x)}{|x|}}.$$

1.2 Examples

The following are explicit examples of strongly admissible weights. Here and throughout, $\exp_k$ denotes the kth iterated exponential.

a) Symmetric exponential weights on the line of polynomial decay:

$$w_\alpha(x) := \exp\left(-|x|^\alpha\right),\ \alpha > 1, \qquad x \in (-\infty, \infty); \tag{1.3}$$

b) Nonsymmetric exponential weights on the line with varying rates of polynomial and faster than polynomial decay:

$$w_{k,l,\alpha,\beta}(x) := \exp(-Q_{k,l,\alpha,\beta}(x))$$

with

$$Q_{k,l,\alpha,\beta}(x) := \begin{cases} \exp_l\left(x^\alpha\right) - \exp_l(0), & x \in [0,\infty), \\ \exp_k\left(|x|^\beta\right) - \exp_k(0), & x \in (-\infty, 0) \end{cases} \tag{1.4}$$

where $l, k \geq 1$ and $\alpha, \beta > 1$;

c) Nonsymmetric exponential weights on $(-1,1)$ with varying rates of decay near ± 1:

$$w^{k,l,\alpha,\beta}(x) := \exp(-Q^{k,l,\alpha,\beta}(x))$$

with

$$Q^{k,l,\alpha,\beta}(x) := \begin{cases} \exp_l(1-x^2)^{-\alpha} - \exp_l(1), & x \in [0,1), \\ \exp_k(1-x^2)^{-\beta} - \exp_k(1), & x \in (-1,0), \end{cases} \tag{1.5}$$

where $l, k \geq 1$ and $\alpha, \beta > 1$.

1.3 Remarks

(a) Definitions 1.1(A-C) define a very general class of possibly nonsymmetric exponential weights of minimal smoothness and with varying rates of decrease on $(c, 0)$ and $[0, d)$. The weights given by (a) are widely called "Freud weights" and are characterized by one rate of polynomial decay at $\pm\infty$. In this case $T \sim 1$ in Definition 1.1B(c). Because of their current popularity, we shall henceforth adopt the name Freud weight in what follows. However, we also allow nonsymmetric weights of polynomial and faster than polynomial decay at c and/or d respectively. Notice that our definition allows w to decrease with one rate on $[0, d)$, and with another on $(c, 0)$. For a detailed perspective on this class of weights and its applications to orthogonal polynomials and various weighted approximation problems of current interest, we refer the reader to [5,6,7,8,9,12,13,15,20,22,23, 24,25,28,29,31] and the many references cited therein.

(b) Definitions 1.1B (a-d) involve smoothness and regularity conditions on w which suffice for uniform bounds on the weighted Hilbert transform, see Theorem 1.6(A-B) below. Note that when w decays as a polynomial, $T \sim 1$, whereas otherwise T increases without bound. Definition 1.1B(d) is needed to control the behavior of Q' close to c and d when Q' grows very quickly. It is trivially satisfied when Q' grows as a polynomial.

(c) The definition of a strongly admissible weight, see Definition 1.1C, is necessary in proving Theorems 1.2, 1.3, 1.5 and 1.7. Here, we need bounds on p_n and estimates for the spacing of its zeroes, which in turn require additional smoothness and regularity assumptions on Q'. Definition 1.1C(b) is a local lip 1/2 condition on Q', and appeared first in [22]. Notice that we do not require Q'' to exist everywhere. Definition 1.1C(d) first appeared in [5,15], although it is motivated by a much older growth lemma of E. Borel. We apply it heavily in the proof of Theorem 1.3 below.

We need some additional notation: If w is strongly admissible and even, it is well known, see [22], that the asymptotic behavior of the recurrence coefficients A_n is expressed in terms of the scaled endpoints of the support of the equilibrium measure for w which is one interval. It is also well known, see [22,24,29,31] and the references cited therein, that firstly these scaled endpoints can be explicitly calculated and are given as the positive roots of the equation

$$u = \frac{2}{\pi} \int_0^1 \frac{a_u t Q'(a_u t)}{\sqrt{1 - t^2}} dt,$$

and secondly that

$$\lim_{n \to \infty} \frac{A_n}{a_n} = 1/2$$

and

$$\lim_{n\to\infty} \frac{A_{n+1}}{A_n} = 1.$$

Here, the number a_u is, as a real valued function of u, uniquely defined and strictly increasing in $(-\infty, \infty)$ with

$$\lim_{u\to-\infty} a_u = c, \qquad \lim_{u\to\infty} a_u = d.$$

We refer the interested reader to [11,14,16,31] and the references cited therein for recent and related analogues of the above circle of ideas for weights whose equilibrium measure is supported on more than one interval.

1.4 Mean convergence of orthonormal expansions

We are ready to state our first result.

Theorem 1.2. *Let $I = (-\infty, \infty)$, w be a symmetric strongly admissible Freud weight, and let $1 < p < \infty$. Suppose $b, B \in I$ satisfy*

$$b < 1 - 1/p, \qquad B > -1/p, \qquad b \leq B.$$

In addition suppose that if $p < 4/3$ then (I):

$$a_n^{\max\{b,-1/p\}-B} n^{1/6(4/p-3)} C_B = O(1),$$

(II): if $p = 4/3$ or 4 then $b < B$, and if $p > 4$ then

$$a_n^{b-\min\{B,1-1/p\}} n^{1/6(1-4/p)} C_B = O(1).$$

Suppose also that

$$\frac{A_{n+1}}{A_n} = 1 + O\left(\frac{1}{n}\right), \qquad n \to \infty. \tag{1.6}$$

Then there exists an infinite subsequence $n = n_j$ of natural numbers such that for $j \geq C$ and for all f satisfying (1.2)

$$||S_n[f]wu_b||_{L_p(I)} \leq C||fwu_B||_{L_p(I)}. \tag{1.7}$$

Moreover, if (1.7) *holds for some infinite subsequence n_j and real b and B, then necessarily the following hold:*

$$b < 1 - 1/p, \qquad B > -1/p, \qquad b \leq B.$$

If $p < 4/3$ then (I):

$$a_n^{\max\{b,-1/p\}-B} n^{1/6(4/p-3)} C_B = O(1),$$

(II): If $p = 4/3$ or 4 then $b < B$, and if $p > 4$ then

$$a_n^{b-\min\{B,1-1/p\}} n^{1/6(1-4/p)} C_B = O(1).$$

In particular, given $\delta > 1$, we have

$$\lim_{j\to\infty} ||(S_n[f] - f)wu_b||_{L_p(I)} = 0 \tag{1.8}$$

for all continuous f with

$$\lim_{|x|\to\infty} |fwu_{B+\delta}|(x) = 0.$$

In [20, Theorem 1.2], Theorem 1.2 is established for every $n \geq 1$ assuming in addition to (1.6) the assumption that

$$\frac{A_n}{a_n} = \frac{1}{2}\left[1 + O\left(\frac{1}{n^{2/3}}\right)\right], \qquad n \to \infty. \tag{1.9}$$

For the Hermite weight, [20, Theorem 1.2] essentially appears in earlier papers of Askey and Waigner, see [1] and Muckenhoupt, see [26] and [27]. Interesting generalizations of the work of Muckenhoupt in [26] and [27] have also been proved by Mhaskar and Xu in [25]. For a detailed discussion of the conditions (1.6) and (1.9), we refer the reader to Remark 1.4 below.

In order to establish Theorem 1.2, we use the following theorem of independent interest.

Theorem 1.3. *Let w be strongly admissible and symmetric. If*

$$\frac{A_{n+1}}{A_n} = 1 + O\left(\frac{1}{(nT(a_n))^{2/3}}\right), \qquad n \to \infty \tag{1.10}$$

then there exists a subsequence $n = n_j$ of natural numbers such that

$$\frac{A_n}{a_n} = \frac{1}{2}\left[1 + O\left(\frac{1}{(nT(a_n))^{2/3}}\right)\right], \qquad j \to \infty. \tag{1.11}$$

The following remark suffices:

Remark 1.4.

(a) Observe that (1.11) is a strong asymptotic whereas (1.10) is a ratio asymptotic. Thus applying the well-known identity

$$\left|\frac{a_{n+1}}{a_n} - 1\right| \sim \frac{1}{nT(a_n)},$$

see [22], it is clear that (1.11) implies (1.10) for every $n \geq 1$. It is the other direction which is new and nontrivial for strongly admissible weights w.

(b) Fortunately, the hypotheses on the recurrence coefficients given by (1.6) and (1.10) are not always vacuous ones. We list some related results on the line, and refer the interested reader to [7,16,21,22,23,28] and the references cited therein for further references and insights.

(A) Let $w = \exp(-Q)$, where Q is an even polynomial of fixed degree. Then (see [16]),

$$\frac{A_n}{a_n} = \frac{1}{2}\left[1 + O\left(\frac{1}{n^2}\right)\right], \qquad n \to \infty.$$

(B) Let $w = W\exp(-Q)$, where Q is an even polynomial of fixed degree with nonnegative coefficients and $W(x) = |x|^\rho$ for some real ρ greater than -1. Then (see [7] and [23]),

$$\frac{A_n}{a_n} = \frac{1}{2}\left[1 + O\left(\frac{1}{n}\right)\right], \qquad n \to \infty.$$

Note that when $\rho \neq 0$, w is not always admissible.

(C) Let $w = w_\alpha$ be given by (1.3). Then (see [21]),

$$\frac{A_n}{a_n} = \frac{1}{2}\left[1 + O\left(\frac{1}{n}\right)\right], \qquad n \to \infty.$$

(D) Let $m \geq 1$ and $w = Ww_{1,1,2m,2m}$ given by (1.4). Then (see [7]),

$$\frac{A_n}{a_n} = \frac{1}{2}\left[1 + O\left(\frac{T(a_n)}{n}\right)\right], \qquad n \to \infty.$$

As a consequence of Theorem 1.3, we now state

Theorem 1.5. *Let $I = (-\infty, \infty)$, w be a strongly admissible symmetric Freud weight and assume that the recurrence coefficients A_n satisfy* (1.6). *Then there exists N_0 and a infinite set of natural numbers Ω such that*

$$\sup_{x\in I} |p_{n+1}(x) - p_{n-1}(x)|\, w(x) \times \left\{\left|1 - \frac{|x|}{a_n}\right| + n^{-2/3}\right\}^{-1/4} \sim a_n^{-1/2} \tag{1.12}$$

for all $1 \leq n \leq N_0$ and $n \geq N_0$, $n \in \Omega$.

The importance of (1.12) lies in the fact that for $|x|$ close to a_n, it improves the known bound (see [22])

$$\sup_{x\in I} |p_n(x)|\, w(x) \left\{\left|1 - \frac{|x|}{a_n}\right| + n^{-2/3}\right\}^{1/4} \sim a_n^{-1/2} \tag{1.13}$$

by a factor of 1/4 as it should. Under the additional assumption (1.9), (1.12) is [20, Theorem 1.1] for $n \geq 1$.

We turn our attention to uniform convergence of orthonormal expansions for strongly admissible symmetric Freud weights on $(-\infty, \infty)$. To this end, we will need bounds on weighted Hilbert transforms. Define formally for continuous $f : I \to (-\infty, \infty)$

$$H[f](x) := \lim_{\varepsilon \to 0^+} \int_{|x-t| \geq \varepsilon} \frac{f(t)}{x-t} dt$$

where the integral above is understood as a Cauchy-Principal valued integral. It is known, see [26,27], that if $b < 1 - 1/p$, $B > -1/p$, $b \leq B$ and $1 < p < \infty$, we have

$$||H[f]u_b||_{L_p((-\infty,\infty))} \leq C||fu_B||_{L_p((-\infty,\infty))}, \tag{1.14}$$

provided the right hand side of (1.14) is finite. Indeed, relations such as (1.14) are essential in studying convergence of orthonormal expansions. This is mainly due to the following identity which follows from the Christoffel-Darboux formula for orthonormal polynomials:

$$S_n[f] = A_n \{p_n H[fp_{n-1}] - p_{n-1} H[fp_n]\}. \tag{1.15}$$

For uniform convergence of orthonormal expansions, it thus seems natural to look for L_∞ analogues of (1.14), but until recently, even for finite intervals, such analogues have been scarce in the literature. We mention that uniform bounds for weighted Hilbert transforms are also important for the numerical solution and stability of integral equations on finite and infinite intervals, see [10] and the references cited therein. To this end, we now state two theorems which hold for any admissible weight on I. We refer the reader to [2,3,4,8,9] and the references cited therein for related results. We recall that for $h > 0$,

$$\Delta_h^1(f, I)(x) := f(x + h/2) - f(x - h/2),\ x \pm h/2 \in I \tag{1.16}$$

is the first symmetric difference operator of f. Then we have:

Theorem 1.6A. *Let $f : I \to (-\infty, \infty)$ be measurable, w be admissible, $B, \varepsilon > 0$ and suppose $||fw(1 + |Q'|)^B||_{L_\infty(I)} < \infty$ and $\frac{\|w\Delta_u^1(f,I)\|_{L_\infty(I)}}{u} \in L_1[0, \varepsilon]$. Then*

$$\|H[fw]\|_{L_\infty(I)} \leq C \left[\|fw(1 + |Q'|)^B\|_{L_\infty(I)} + \int_0^\varepsilon \frac{\|w\Delta_u^1(f, I)\|_{L_\infty(I)}}{u} du \right]. \tag{1.17}$$

Theorem 1.6B. *Let $f : I \to (-\infty, \infty)$ be differentiable, w be admissible, $B > 0$ and suppose $||fw(1 + |Q'|)^B||_{L_\infty(I)} < \infty$ and $||f'w||_{L_\infty(I)} < \infty$. Then*

$$\|H[fw]\|_{L_\infty(I)} \leq C \left[\|fw(1 + |Q'|)^B\|_{L_\infty(I)} + \|f'w\|_{L_\infty(I)}\right]. \tag{1.18}$$

Note the factor $(1 + |Q'|)^B$ on the right-hand side of (1.17) and (1.18). For Q even and of polynomial growth, (the case $T \sim 1$), we may take it to be essentially u_{B_1} for some $B_1 > 0$. Theorems 1.6(A-B) improve [9, Theorem 1.1] and [4, Theorem 1] in several respects. Firstly they replace the weighting factor of w^2 in [9] by the correct factor $w(1+|Q'|)^B$. This later observation is crucial in the formulation of Theorem 1.7 below. Secondly, Theorems 1.6(A-B) hold simultaneously for a much larger class of possibly nonsymmetric weights with varying rates of decay on $(c, 0)$ and $[0, d)$. Using Theorems 1.6(A-B), we are able to announce:

Theorem 1.7. *Let $I = (-\infty, \infty)$, w be a strongly admissible symmetric Freud weight, $B > 0$, assume (1.6) and further that*

$$a_n^{b-\min\{B,1\}} n^{1/6} C_B = O(1),\ n \geq 1$$

where C_B is 1 if $B \neq 1$ and $\log n$ if $B = 1$. Let $f : I \to (-\infty, \infty)$ be differentiable and suppose that $f(t)(t - x)$ has fixed sign in $[x - \varepsilon, x + \varepsilon]$ for some small and positive ε. Suppose finally that $||fwu_B||_{L_\infty(I)} < \infty$ and $||(fw)'||_{L_\infty[x,x+\varepsilon]} < \infty$. Then there exists an infinite subsequence $n = n_j$ of natural numbers such that for all $j \geq C$

$$|S_n[f]w|(x) \leq C \left[||fwu_B||_{L_\infty(I)} + ||(fw)'||_{L_\infty[x,x+\varepsilon]}\right]. \tag{1.19}$$

If f' is also continuous and

$$\lim_{|x|\to\infty} |fQ'wu_{B+\delta}|(x) = 0$$

for some $\delta > 1$, then

$$\lim_{j\to\infty} |(S_n[f] - f)w|(x) = 0. \tag{1.20}$$

If we assume (1.9), then (1.20) holds for every $n \geq 1$.

Theorem 1.7 provides sufficient conditions for uniform convergence of orthogonal expansions for strongly admissible Freud weights on the line. Its proof will appear in a future paper.

The remainder of this paper is devoted to the proofs of Theorem 1.2, Theorem 1.3, Theorem 1.5 and Theorems 1.6(A-B).

§2. Proofs of Theorems 1.2, 1.3, 1.5 and 1.6(A-B)

In this section, we prove Theorems 1.2, 1.3, 1.5 and 1.6(A-B).

2.1 The Proof of Theorem 1.3.

Let $n \geq C$, and define

$$m = m(n) = [n^{1/3}(T(a_n))^{1/3}]$$

where $[x]$ denotes the greatest integer $\leq x$. It follows using Definition 1.1C (a) and Remark 1.4(a) that uniformly for $r = 1, ..., m$ and $n \geq 1$,

$$T(a_{n+r}) \sim T(a_n).$$

Armed with this identity, we apply (1.10) repeatedly and deduce that there exists $N = N(m)$ and $D > 0$ such that for $n \geq N$,

$$A_{n+r} \geq \left(1 - \frac{D}{(nT(a_n))^{2/3}}\right) A_n, \qquad r = 1, 2, ..., m. \tag{2.1}$$

Here $D > 0$ does not depend on n or m so we fix it. Now set

$$\varepsilon = \varepsilon(n) = \frac{D}{(nT(a_n))^{2/3}}.$$

A careful adaption of the proof of [18, Theorem 6] then shows that

$$\begin{aligned} x_{1,n}(w) &\geq 2(1-\varepsilon)A_n \cos\frac{\pi}{m+1} \\ &\geq 2(1-\varepsilon)A_n(1 - C/m^2) \\ &\geq 2\left(1 - \frac{D}{(nT(a_n))^{2/3}}\right) A_n \left(1 - \frac{C}{(nT(a_n))^{2/3}}\right). \end{aligned} \tag{2.2}$$

Now recall that

$$\left|\frac{x_{1,n}}{a_n} - 1\right| = O\left(\frac{1}{n^{2/3}T(a_n)^{2/3}}\right). \tag{2.3}$$

Thus (2.2) and (2.3) give

$$\frac{A_n}{a_n} \leq \frac{1}{2}\left[1 + O\left(\frac{1}{(nT(a_n))^{2/3}}\right)\right], \qquad n \geq C. \tag{2.4}$$

Another careful inspection of [18, Theorem 7], reveals that we have

$$\lim_{n\to\infty} \frac{x_{1,n}}{\max_{k\leq n}\alpha_k} \leq 2.$$

Thus, we may apply this with (2.3) to obtain

$$\frac{\max_{k\leq n} A_k}{a_n} \geq \frac{1}{2}\left(1 - \frac{1/C}{(nT(a_n))^{2/3}}\right). \tag{2.5}$$

Choosing an increasing sequence n_r with

$$\max_{k\leq n_r} A_k = A_{n_r}, \tag{2.6}$$

and applying (2.4)–(2.6) gives the theorem.

2.2 Proof of Theorems 1.2 and 1.5.

We sketch the important ideas of the proof. The remaining technical details are very similar to [20, Theorem 1.1], and so we refer to reader to that paper for these.

Step 1: We reduce the proof of Theorem 1.5 to one important case.

(a) By symmetry we may assume that $x \geq 0$.

(b) It suffices to prove (1.12) for n sufficiently large.

(c) Using infinite finite inequalities it suffices to prove (1.12) for

$$0 \leq x \leq a_n \left(1 - \frac{C}{n^{2/3}}\right).$$

(d) For $0 \leq x \leq a_{n/2}$,

$$\left|1 - \frac{|x|}{a_n}\right| + n^{-2/3} \sim 1$$

so (1.12) follows from (1.13).

Step 2: Without loss of generality we may thus assume that

$$a_{n/2} \leq x \leq a_n \left(1 - \frac{C_1}{n^{2/3}}\right)$$

for some $C_1 > 0$ which will be chosen later. Let us define for $y \geq 0$

$$\tau_n(y) := \frac{1}{A_n^2} \sum_{k=0}^{n-1} (\alpha_{k+1}^2 - \alpha_k^2) p_k^2(y).$$

Then the Dombrowski-Fricke identity, see [17], gives

$$|p_{n+1}(y) - p_{n-1}(y)| w(x) \leq \left\{2\tau_{n+1}(y) w^2(y)\right\}^{1/2} + \left\{2\tau_n(y) w^2(y)\right\}^{1/2}, \tag{2.7}$$

for

$$0 \leq y \leq 2\min\{A_n, A_{n+1}\}.$$

Applying Theorem 1.3, we deduce that there exists $D > 0$ and an infinite set of natural numbers Ω such that (2.7) holds for all

$$0 \leq y \leq a_n \left(1 - Dn^{-2/3}\right), \qquad n \in \Omega.$$

Choose $C_1 = D$ in the above and assume, as we may, that (2.7) holds for x and $n \in \Omega$.

Step 3: Estimation of $\tau_n(x)$ for $n \in \Omega$: We write

$$\tau_n(x) := \frac{1}{A_n^2}\left\{\sum_{k=0}^{[n/4]} + \sum_{k=[n/4]+1}^{n-1}\right\}(\alpha_{k+1}^2 - \alpha_k^2)p_k^2(x)$$

$$= \tau_{n,1}(x) + \tau_{n,2}(x).$$

Now applying (1.6), we have that

$$\tau_{n,2}(x)w^2(x) \leq C\frac{1}{n}w^2(x)\lambda_n(x)^{-1},$$

where λ_n are the Cotes numbers for w^2. But for this range of x, it is well known that

$$\lambda_n(x)^{-1}w^2(x) \sim \frac{n}{a_n}\sqrt{\left|1 - \frac{|x|}{a_n}\right| + n^{-2/3}}.$$

Thus,

$$\tau_{n,2}(x)w^2(x) \leq Ca_n^{-1}\sqrt{\left|1 - \frac{|x|}{a_n}\right| + n^{-2/3}}.$$

Moreover, using infinite-finite range inequalities yields

$$\tau_{n,1}(x)w^2(x) = O\left(\exp(-Cn)\right).$$

Combining the above estimates for both $\tau_{n,1}$ and $\tau_{n,2}$ yields Theorem 1.5. Theorem 1.2 then follows using [20, Theorem 1.2], Theorem 1.5 and Theorem 1.3 setting $n = n_j$. The crucial point in the proofs of both Theorem 1.2 and Theorem 1.5 is the removal of the strong asymptotic (1.9), and this is achieved by means of Theorem 1.3.

2.3 The Proof of Theorems 1.6(A-B).

We begin with Theorem 1.6(A). We may assume that $x \in [0, d)$. The other case is similar. We consider several subcases. We suppose first that $d = \infty$, $c = -\infty$, and that x is sufficiently close to d. More precisely, let us choose a constant $D > 0$ so close to d so that for $D < x < d$

$$\frac{1}{Q'(x+\varepsilon)} < \varepsilon.$$

(We will in practice always take ε small since clearly if Theorem 1.6(A-B) holds for such ε, then it holds for larger ε). Fix x and define

$$A(x) = A_\varepsilon(x) := \frac{1}{2Q'(x+\varepsilon)}.$$

Note $|Q'|(y) = Q'(y)$ for y close enough to d, and $|Q'|(y) = -Q'(y)$ for y close enough to c. This follows from the definition of T and Definitions 1.1(B-C). Let us write

$$\begin{aligned} H[fw](x) &= \left(\int_{|t|>2x} + \int_{-2x}^{0} + \int_{0}^{2x}\right) \frac{f(t)w(t)}{t-x}dt \\ &= I_1(x) + I_2(x) + I_3(x). \end{aligned} \tag{2.8}$$

We first estimate $I_1(x)$. Here

$$|t-x| \geq |t| - |x| \geq |t| - |t|/2 = |t|/2$$

so

$$\begin{aligned} |I_1(x)| &\leq 2\int_{|t|>2x} \frac{|f(t)|w(t)}{|t|}dt \\ &\leq C\|fw(1+|Q'|)^B\|_{L_\infty(I)} \int_{|t|\geq 2} \frac{1}{|t|(1+|Q'|(t)|)^B}dt \\ &\leq C\|fw(1+Q')^B\|_{L_\infty(I)}. \end{aligned}$$

Similarly,

$$\begin{aligned} |I_2(x)| &= \left|\int_{-2x}^{0} \frac{f(t)w(t)}{t-x}dt\right| \leq \|fw(1+|Q'|)^B\|_{L_\infty(I)} \int_{-2x}^{0} \frac{1}{x-t}dt \\ &\leq C\|fw(1+|Q'|)^B\|_{L_\infty(I)} \int_{x}^{3x} \frac{du}{u} \\ &\leq C\|fw(1+|Q'|)^B\|_{L_\infty(I)}. \end{aligned}$$

We proceed with $I_3(x)$. Note that by choice of $A(x)$ and using [22, Lemma 3.2a],

$$w(y) \sim w(x)$$

uniformly for every $y \in I$ with $|x-y| \leq 2A(x)$. (The latter lemma implies that $Q(s) \geq Q(r)$, $s/r \geq 1$.) We then split $I_3(x)$ as follows: Write for $\beta := \beta(\varepsilon) > 0$,

$$\begin{aligned} I_3(x) &= \int_0^{2x} \frac{f(t)w(t)}{t-x}dt \\ &= \left(\int_0^{x/\beta} + \int_{x/\beta}^{x-\varepsilon} + \int_{x-\varepsilon}^{x-A(x)} + \int_{x-A(x)}^{x+A(x)} + \int_{x+A(x)}^{2x}\right) \frac{f(t)w(t)}{t-x}dt \\ &= I_{31}(x) + I_{32}(x) + I_{33}(x) + I_{34}(x) + I_{35}(x). \end{aligned}$$

Then,

$$
\begin{aligned}
|I_{31}(x)| &\le C\|fw(1+|Q'|)^B\|_{L_\infty(I)} \int_0^{x/\beta} \frac{(1+|Q'|(t))^{-B}}{x-t}dt \\
&\le C\|fw(1+|Q'|)^B\|_{L_\infty(I)} \int_0^{x/\beta} \frac{1}{x-t}dt \\
&\le C\|fw(1+|Q'|)^B\|_{L_\infty(I)}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|I_{32}(x)| &\le C\|fw(1+|Q'|)^B\|_{L_\infty(I)} \frac{\log x}{(1+Q'(x/\beta))^B} \\
&\le C\|fw(1+|Q'|)^B\|_{L_\infty(I)}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
|I_{33}(x)| &\le C\|fw(1+|Q'|)^B\|_{L_\infty(I)} \int_{x-\varepsilon}^{x-A(x)} \frac{(1+|Q'|(t))^{-B}}{x-t}dt \\
&\le C\|fw(1+|Q'|)^B\|_{L_\infty(I)} \frac{\log Q'(x+\varepsilon)}{(1+Q'(x-\varepsilon))^B} \\
&\le C\|fw(1+|Q'|)^B\|_{L_\infty(I)}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&|I_{35}(x)| = \left| \int_{x+A(x)}^{2x} \frac{f(t)w(t)}{t-x}dt \right| \\
&\le C\|fw(1+|Q'|)^B\|_{L_\infty(I)}(1+Q'(x+A(x)))^{-B} \int_{A(x)}^{x} \frac{1}{u}du \\
&\le C\|fw(1+|Q'|)^B\|_{L_\infty(I)}(1+Q'(x+A(x)))^{-B}(\log Q'(x+\varepsilon)+\log x) \\
&\le C\|fw(1+|Q'|)^B\|_{L_\infty(I)}.
\end{aligned}
$$

Finally we consider $I_{34}(x)$. We write

$$
\begin{aligned}
&|I_{34}(x)| = \left| \int_{x-A(x)}^{x+A(x)} \frac{f(t)w(t)}{t-x}dt \right| \\
&\le \left| \int_{x-A(x)}^{x+A(x)} f(t)\frac{w(t)-w(x)}{t-x}dt \right| + w(x) \left| \int_{x-A(x)}^{x+A(x)} \frac{f(t)-f(x)}{t-x}dt \right| \\
&= |I_{341}(x)| + |I_{342}(x)|.
\end{aligned}
$$

We begin by making the substitution $t = u/2 + x$ into $I_{342}(x)$. Then we have

$$\begin{aligned}
|I_{342}(x)| &\le Cw(x)\int_0^{2A(x)} \left|\frac{f(x+u/2)-f(x-u/2)}{u}\right| du \\
&\le C\int_0^{2A(x)} \left\|\Delta_u^1(f,I)w\right\|_{L_\infty(I)} \frac{1}{u}du \\
&\le C\int_0^{\varepsilon} \left\|\Delta_u^1(f,I)w\right\|_{L_\infty(I)} \frac{1}{u}du.
\end{aligned}$$

Also

$$\begin{aligned}
&|I_{341}(x)| = \left|\int_{x-A(x)}^{x+A(x)} f(t)\frac{w(t)-w(x)}{t-x}dt\right| \\
&\le C\|fw(1+|Q'|)^B\|_{L_\infty(I)} \int_{x-A(x)}^{x+A(x)} w^{-1}(t)|w'(\eta)|dt \\
&\le CA(x)w(x-A(x))w^{-1}(x+A(x))Q'(x+A(x))\|fw(1+|Q'|)^B\|_{L_\infty(I)} \\
&\le C\|fw(1+|Q'|)^B\|_{L_\infty(I)}.
\end{aligned}$$

We observe that if $1 \le x < D$, then the estimates for $I_1(x)$ go through as before as we only needed the fact that $D > 1$ to ensure that the integral converged for t much larger then x. $I_2(x)$ follows without change except that we use the boundedness of $|Q'|$ rather than its sign. For $I_3(x)$, x is bounded and $w \sim 1$, so the proof is easier than before. If $0 < x < 1$, then we write

$$H[f;w](x) = \int_{-\infty}^{\infty} \frac{f(t)w(t)}{t-x}dt = \left(\int_{-\infty}^{x-1} + \int_{x-1}^{x+1} + \int_{x+1}^{\infty}\right) \frac{w(t)f(t)}{t-x}dt.$$

For the first two integrals, t is bounded away from x, and for the third we proceed as above, but the proof is easier since x is bounded and $w \sim 1$. Suppose now that d and c are finite. Let us first suppose that x is close to d. Then choose $D > 0$ such that $D \le x < d$, and write

$$H[f;w](x) = \left(\int_c^0 + \int_0^d\right) \frac{w(t)f(t)}{t-x}dt.$$

For the first integral, $t-x$ is bounded away from 0, so bounding this term gives us the required estimate. For the second integral, split as in $I_3(x)$. (Note that here we choose ε at the start small enough so that $x \pm \varepsilon \in I$). A very similar and easier argument works if also $\gamma \le x \le D$ for some fixed and positive γ, for here $w \sim 1$. If for this γ, $0 \le x < \gamma$, then split

$$H[f;w](x) = \int_c^d \frac{f(t)w(t)}{t-x}dt = \left(\int_c^{x-\gamma} + \int_{x-\gamma}^{x+\gamma} + \int_{x+\gamma}^{d}\right) \frac{w(t)f(t)}{t-x}dt.$$

Then the estimate goes through very much as above. Finally, suppose that d is finite and $c = -\infty$. (The other case is similar). Then choose $A > 1$, and write

$$H[f;w](x) = \left(\int_{-\infty}^{-Ax} + \int_{-Ax}^{0} + \int_{0}^{d} \right) \frac{w(t)f(t)}{t-x} dt.$$

This proves Theorem 1.6(A). Theorem 1.6(B) follows by noting that in the proof of Theorem 1.6(A), for a given x, we may take ε small enough so that $w(x \pm u/2) \sim w(x)$ for all $u \le \varepsilon$. This easily yields the result.

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