Weighted Polynomials on Discrete Sets

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Abstract

For a real interval I of positive length, we prove a necessary and sufficient condition which ensures that the *continuous* $L_p(0 < p \leq \infty)$ norm of a weighted polynomial, $P_n w^n$, $\deg P_n \leq n$, $n \geq 1$ is in an nth root sense, controlled by its corresponding *discrete* Hőlder norm on a very general class of discrete subsets of I. As a by product of our main result, we establish Nikolśkii inequalities and theorems dealing with zero distribution, zero location and sup and L_p infinite-finite range inequalities.

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1 Introduction and Statement of Results

Let I be a real interval of positive length. A weighted polynomial of degree at most $n \geq 1$ on I is an expression of the form $P_n w^n$ where P_n is an algebraic polynomial of degree at most $n \geq 1$ and

$$w : I \to [0, \infty) \tag{1.1}$$

is a positive, non identically zero continuous weight on I. Throughout, $Q := -\log w$ will be the external field induced by the weight w. If I is unbounded, we suppose further that

$$\lim_{|x|\to\infty} |x| w^{1-\eta}(x) = 0,\ x \in I \tag{1.2}$$

for some $0 < \eta < 1$. In this paper, we obtain a necessary and sufficient condition which ensures that the continuous $L_p(0 < p \leq \infty)$ norm of a weighted polynomial, $P_n w^n$, $\deg P_n \leq n$, $n \geq 1$ is in an nth root sense, controlled by its corresponding discrete Hőlder norm on a very general class of discrete subsets of I. As a consequence, we generalize a theorem of Kuijlaars and Van Assche, [9, Theorem 7.2] and deduce sharp Nikolśkii inequalities as well as theorems dealing with zero distribution, zero location and infinite-finite range inequalities. The problem of studying asymptotics of weighted polynomials in discrete L_p norms was initiated by Rakhmanov in [10] and has recently been investigated further by Dragnev and Saff in [6], Kuijlaars and Van Assche in [9] and Beckermann in [1].

1.1 Background

To formulate our main results, we require some needed notation and quantities:

Throughout Π_n will denote the class of polynomials of degree at most $n \geq 1$, Π_n^* the class of monic polynomials of degree n, $n \geq 1$ and

$$E_n := \{\eta_{1,n} < ... < \eta_{n,n}\}_{n=1}^{\infty}$$

a triangular scheme of points in I. If I is unbounded, we will suppose henceforth that the points of E_n have no finite points of accumulation. Define, for each n, the Hőlder function space:

$$L_{p,H}(E_n) := \left\{f : E_n \longrightarrow \mathbb{R} | \, \|f\|_{L_{p,H}(E_n)} < \infty\right\}$$

where,

$$\|f\|_{L_{p,H}(E_n)} := \begin{cases} \sup_{x \in E_n} |f|(x), & p = \infty \\ \left(\sum_{x \in E_n} |f|^p(x)\right)^{\frac{1}{p}}, & 0 < p < \infty. \end{cases}$$

Moreover, for a measurable subset $E \subseteq I$, denote by $L_p(E)$, the usual continuous L_p function space for any $0 < p \leq \infty$. Throughout, C will denote a positive constant independent of n and P_n which may take on different values at different times.

A crucial tool in our analysis will be the concept of an *equilibrium measure*. Given a Borel measure μ on I, its weighted energy is given by

$$I_w(\mu) := \int \int \log \frac{1}{|s-t|} d\mu(s) d\mu(t) - 2 \int \log w(t) d\mu(t).$$

The *equilibrium measure* in the presence of the weight w, is the unique Borel probability measure μ_w on I minimizing the weighted energy among all probability measures. Thus

$$I_w(\mu_w) = \min\{I_w(\mu) : \mu \in \mathcal{P}(I)\}$$

where $\mathcal{P}(I)$ denotes the class

$$\mathcal{P}(I) := \{\mu : \mu \text{ is a Borel probability measure on } I\}.$$

Discrete sets A triangular scheme E_n will be called *admissible* in I, if the following conditions below hold:

Distribution Condition A

For each compact $A \subseteq I$, $\sigma_n(A)$, $n = 1, 2, ...$ is finite, where

$$\sigma_n(A) := \frac{1}{n} \text{ card } (A \cap E_n)$$

denotes the normalized counting measure of E_n. Moreover, suppose there is a Borel measure σ with support I and total mass > 1 satisfying that for every compact $K \subset I$, the restricted measure $\sigma|_K$ has a continuous logarithmic potential $U^{\sigma|_K}$ and

$$\lim_{n\to\infty} \int f d\sigma_n = \int f d\sigma$$

for all continuous f on I with compact support. Such a measure, if it exists, will be called an *admissible constraint*.

Separation Condition B

Let $\lambda_w^\sigma \leq \sigma$ be the unique probability measure which minimizes the energy

$$I_w(\mu) := \int \int \log \frac{1}{|s-t|} d\mu(s) d\mu(t) - 2 \int \log w(t) d\mu(t) \tag{1.3}$$

over all Borel probability measures μ where the difference $\sigma - \mu$ is positive on I and let I_0 be a bounded interval with $\text{supp}(\lambda_w^\sigma) \subseteq I_0 \subseteq I$. Consider the polynomial

$$R_n(x) := \prod_{\eta_{i,n} \in I_0} (x - \eta_{i,n}),\ x \in I$$

and let $\sigma_1 := \sigma|_{I_0}$. Suppose that for q.e. $\eta \in I_0$,

$$|R_n'(\eta_{k,n})|^{1/n} \to \exp(-U^{\sigma_1}(\eta)) \tag{1.4}$$

as $n \to \infty$ whenever

$$\eta_{k,n} \to \eta,\ k = k(n).$$

Here by q.e, we mean with the exception of a set of logarithmic capacity zero.

Condition C to control the discrete L_p norm of $P_n w^n$ from far away points

Assume that for all $\varepsilon > 0$ and $0 < p < \infty$

$$\limsup_{n\to\infty} \| \left(x^{1+\varepsilon} w(x)\right)^n \|_{L_{p,H}(E_n)}^{1/n} < \infty. \tag{1.5}$$

We find it instructive to present a short remark dealing with the generality of our discrete sets defined above with natural examples. This is contained in Remark A below. None of the statements in Remark A are used in our proofs and so the reader may read this remark independently of the rest of this paper.

Remark A: Discrete sets.

(a) It is true that the following stronger separation condition implies Condition B:

Assume that there exists $\rho > 0$ with

$$\min_i |\eta_{i+1,n} - \eta_{i,n}| \geq \frac{\rho}{n} \tag{1.6}$$

Secondly, Condition B implies that if I is bounded, then an admissible triangular array in I may be taken as the zeros of any system of orthogonal polynomials with respect to a weight $W > 0$ a.e. on I with all moments

$$\int_I x^n W(x)dx,\ n = 0, 1, ...$$

finite.

It is also true that Condition B is implied by the relative distance condition proposed by Rakhmanov in [10, Theorem 2] as well as the separation condition of Dragnev and Saff in [6, Definition 3.1]. Finally, in ([8], (8.1)) and [1, pg 4], Beckermann and Rakhmanov have suggested another separation condition which was used extensively in [1]. More, precisely, if I is bounded, Condition B is replaced by Condition R1:

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{x,y\in E_n,\, x\neq y} \log\frac{1}{|x-y|} = I(\sigma) < \infty$$

and if I is unbounded, Condition B is replaced by Condition R2: There exists an open set V with

$$\mathrm{Supp}(\lambda_w^\sigma) \subset V \text{ and } \lim_{n\to\infty} \frac{1}{n^2} \sum_{x,y\in E_n,\, x\neq y} \log\frac{1}{|x-y|} = I(\sigma|V) < \infty.$$

We remark that it can be shown using general principles in potential theory that the separation condition of Dragnev and Saff in [6, Theorem 1.8] implies Rakhmanov's Condition R1 and that the separation condition (1.8) of [9, Definition 3], does not imply Rakhmanov's Condition R2. It is an open question as to the exact relationship between our Condition B and Conditions R1 and R2. Very general points of this type have recently found other important applications in interpolation and numerical integration, see [4] and [5].

(b) The L_p condition on the admissible triangular arrays E_n is needed to control the contribution to our discrete L_p norms from far way points. Its present form with $\varepsilon = 0$ appears in [1, Theorem 1.3] and suffices for the Fekete point results proved in the latter paper. A more restrictive condition to ours can be found in [9, page 208].

1.2 A Weighted Polynomial Inequality

. We shall prove:

Theorem 1.1: A weighted polynomial inequality. *Let $0 < p \leq q \leq \infty$ and let E_n, $n \geq 1$ be an admissible triangular array in I with admissible constraint σ. Then for any sequence of polynomials $P_n \in \Pi_n$*

$$\lim_{n\to\infty} \left(\frac{\|P_n w^n\|_{L_q(I)}}{\|P_n w^n\|_{L_{p,H}(E_n)}} \right)^{1/n} = 1 \tag{1.7}$$

iff

$$\mu_w \leq \sigma. \tag{1.8}$$

Remark B Theorem 1.1 is a basic result and all our other results below depend on it. The formula (1.8) means that the measure $\sigma - \mu_w$ is a positive measure. The sufficiency of (1.8) was first proved by Kuijlaars and Van Assche in [9, Theorem 7.2] under the conditions that $I = (0, \infty)$, $p = \infty$ and for sets E_n that satisfy the stronger separation condition (1.6). The necessity of Theorem 1.1 is new for all the classes of points considered in [10], [6], [9] and [1]. Thus Theorem 1.1 generalizes [9, Theorem 7.2] in two aspects. Firstly it shows that (1.8) is in fact necessary and secondly it works for any real interval I, for any $0 < p \leq \infty$ and for a larger and more general class of sets E_n.

It is instructive at this point to illustrate the usefulness of Theorem 1.1 by means of an example.

1.3 Example 1

Let natural numbers N and n be given and let E_n be any sequence of discrete subsets of N equally spaced points in $[-1, 1]$ with spacing ρ/n for some fixed $\rho > 0$. Moreover, suppose that

$$\lim_{n\to\infty} \frac{N}{n} = \lambda > 1.$$

For the given λ set

$$r := \sqrt{1 - \lambda^{-2}}.$$

In [11], Rakhmanov has asked the following question: Find the largest set $A \subseteq [-1, 1]$ such that for any sequence of polynomials $P_n \in \Pi_n$,

$$\limsup_{n\to\infty} \left(\frac{\|P_n\|_{L_\infty(A)}}{\|P_n\|_{L_{\infty,H}(E_n)}} \right)^{1/n} \leq 1. \tag{1.9}$$

If $\frac{n^2}{N}$ is bounded, then Coppersmith and Rivlin, see [3], proved more than (1.9) namely:

$$\exp\left(\frac{n^2}{CN}\right) \le \left(\frac{\|P_n\|_{L_\infty[-1,1]}}{\|P_n\|_{L_{\infty,H}(E_n)}}\right) \le \exp\left(\frac{Cn^2}{N}\right).$$

Under our different assumptions of n and N we ask what can be said about A? Indeed, using the necessity of Theorem 1.1, it is well known that the equilibrium measure for the interval $[-1,1]$ given by

$$d\mu(x) = \frac{1}{\pi\sqrt{1-x^2}}dx,\ x \in (-1,1),$$

clearly violates (1.8) with σ, the uniform distribution, given by

$$d\sigma(x) = \lambda dx,\ x \in (-1,1).$$

Thus (1.7) cannot hold with $A = [-1,1]$.

1.4 Sharp Nikolśkii inequality

As a consequence of the proof of Theorem 1.1 we will deduce an important Nikolśkii inequality, (see Theorem 2.1 below), which in its sharp form is new even under the weaker conditions of [9, Lemma 8.3(b)].

Nikolśkii Inequality *Assume the hypotheses of Theorem* 1.1.

(a) *Then uniformly for any polynomial* $P_n \in \Pi_n$ *and* $n \ge 1$,

$$1 \le \left(\frac{\|P_n w^n\|_{L_{p,H}(E_n)}}{\|P_n w^n\|_{L_{\infty,H}(E_n)}}\right) \le Cn^{1/p},\ 0 < p < \infty.$$

(b) *Moreover, there exists* $\delta > 0$ *such that uniformly for every polynomial* $P_n \in \Pi_n$ *and* $n \ge 1$

$$1 - \exp(-\delta n) \le \left(\frac{\|P_n w^n\|_{L_{p,H}(E_n)}}{\|P_n w^n\|_{L_{q,H}(E_n)}}\right) \le Cn^{1/p-1/q},\ 0 < p < q < \infty.$$

1.5 Where does the nth root discrete L_p norm of a weighted extremal polynomial live?

In what follows we now describe the size of the largest set $A \in I$ where (1.7) holds without (1.8). We first consider **infinite-finite** range inequalities for weighted discrete polynomials. Their analogues for the continuous case can be found in [12, Theorem 3.6.1].

1.6 Sup norm: Infinite-finite range inequalities

Theorem 1.2 *Assume the hypotheses of Theorem* 1.1 *and for each* $k > 0$, *define*

$$S_k := \left\{x \in I : U^{\lambda_w^\sigma}(x) + Q(x) - F_w^\sigma \leq k\right\} \tag{1.10}$$

where F_w^σ *is the unique constant satisfying the variational inequalities:*

$$U^{\lambda_w^\sigma}(x) - \log w(x) \leq F_w^\sigma,\ x \in \operatorname{supp}(\lambda_w^\sigma) \tag{1.11}$$

and

$$U^{\lambda_w^\sigma}(x) - \log w(x) \geq F_w^\sigma,\ x \in \operatorname{supp}(\sigma - \lambda_w^\sigma). \tag{1.12}$$

(a) *Then for every* $\varepsilon > 0$, *there exists* N_0 *such that for every* $n \geq N_0$ *and for every polynomial* $P_n \in \Pi_n$

$$\begin{aligned} |P_n w^n|(x) &\leq \|P_n w^n\|_{L_{\infty,H}(E_n)} \times \\ &\times \exp\left(-n\left(U^{\lambda_w^\sigma}(x) + Q(x) - F_w^\sigma - \varepsilon\right)\right),\ x \in I\backslash S_\varepsilon. \end{aligned} \tag{1.13}$$

(b) *In particular, we have for every sequence of polynomials* $P_n \in \Pi_n$

$$\limsup_{n\to\infty} \left(\frac{\|P_n w^n\|_{L_\infty(\overline{I\backslash S_0})}}{\|P_n w^n\|_{L_{\infty,H}(E_n)}}\right)^{1/n} \leq 1. \tag{1.14}$$

Theorem 1.2 says that the nth root sup norm of a weighted discrete polynomial essentially lives in the set $\overline{I\backslash S_0}$. The next result says that the size of this set is essentially best possible:

Theorem 1.3 *Assume the hypotheses of Theorem 1.1. For any* $0 < p \leq \infty$, *let* $P_{n,p}^* \in \Pi_n^*$ *be an extremal polynomial with respect to* E_n *and* w *satisfying*

$$\|P_{n,p}^* w^n\|_{L_{p,H}(E_n)} = \inf_{P_n \in \Pi_n^*} \|P_n w^n\|_{L_{p,H}(E_n)}. \tag{1.15}$$

Suppose there exists a set $A \subseteq I$ *satisfying*

$$\overline{I\backslash S_0} \subset A \subseteq I$$

for which

$$\limsup_{n\to\infty} \left(\frac{\|P_n^* w^n\|_{L_\infty(A)}}{\|P_n^* w^n\|_{L_{\infty,H}(E_n)}}\right)^{1/n} \leq 1. \tag{1.16}$$

Then

$$\operatorname{cap}(A\backslash(\overline{I\backslash S_0})) = 0 \tag{1.17}$$

where cap *denotes logarithmic capacity.*

Example 2 If we return to Example 1 above with σ the uniform distribution and $w \equiv 1$, it is known, see [10], that $\overline{I\backslash S_0} = [-r, r]$ where

$$r := \sqrt{1 - \lambda^{-2}}.$$

Thus $[-r, r]$ is the largest set in the sense of (1.17) for which (1.7) holds.

1.7 L_p norm: Infinite-finite range inequalities

We now turn to the question of where the nth root L_p norm of a weighted discrete polynomial lives. Let us set for this purpose:

$$S := \text{supp}(\lambda_w^\sigma) \cap \text{supp}(\sigma - \lambda_w^\sigma).$$

Then as we will show below, the nth root L_p norm of a discrete weighted polynomial is supported on the set S, the free part of the measure λ_w^σ.

Theorem 1.4 *Assume the hypotheses of Theorem* 1.1 *with* $0 < p \leq \infty$.

(a) *Then for every sequence of polynomials* $P_n \in \Pi_n$

$$\limsup_{n\to\infty} \left(\frac{\|P_n w^n\|_{L_p(S)}}{\|P_n w^n\|_{L_{p,H}(E_n)}} \right)^{1/n} \leq 1. \tag{1.18}$$

(b) *Let* $P_{n,p}^* \in \Pi_n^*$ *be an extremal polynomial with respect to* E_n *and* w *given by* (1.15). *Moreover suppose that for* $p = \infty$, S *has positive logarithmic capacity and for* $0 < p < \infty$, S *is the union of a finite number of finite non degenerate intervals. Then*

$$\lim_{n\to\infty} \left(\frac{\|P_{n,p}^* w^n\|_{L_p(S)}}{\|P_{n,p}^* w^n\|_{L_{p,H}(E_n)}} \right)^{1/n} = 1. \tag{1.19}$$

(c) *Finally, suppose that for* $0 < p < \infty$, *we only require* S *to have positive logarithmic capacity, then*

$$\liminf_{n\to\infty} \left(\frac{\|P_{n,p}^* w^n\|_{L_p(N)}}{\|P_{n,p}^* w^n\|_{L_{p,H}(E_n)}} \right)^{1/n} \geq 1 \tag{1.20}$$

where N *is a neighborhood of the larger set* S_0 *given by* (1.10). *We remind the reader that in view of* (1.11) *and* (1.12)

$$S \subseteq \text{supp}(\lambda_w^\sigma) \subseteq S_0.$$

1.8 Zero distribution of weighted discrete extremal polynomials.

The following theorem for an admissible triangular array, is the analogue of [1, Theorem 1.3], which is in turn the analogue of [12, Theorem 3.3.1] for sets of positive logarithmic capacity.

Theorem 1.5 *Let* $0 < p \leq \infty$ *and assume the hypotheses of Theorem* 1.1. *Let* $\nu_n(P_n^*)$ *be the normalized counting measure of the zeros of* $P_{n,p}^*$.

Then the following are true:

(a) *For any sequence of polynomials* $P_n \in \Pi_n^*$

$$\liminf_{n\to\infty} \|P_n w^n\|_{L_{p,H}(E_n)}^{1/n} \geq \exp(-F_w^\sigma). \tag{1.21}$$

Moreover,

$$\lim_{n\to\infty} \|P_{n,p}^* w^n\|_{L_{p,H}(E_n)}^{1/n} = \exp(-F_w^\sigma). \tag{1.22}$$

(b) *For every sequence of polynomials* $P_n \in \Pi_n^*$ *with*

$$\lim_{n\to\infty} \|P_n w^n\|_{L_{p,H}(E_n)}^{1/n} = \exp(-F_w^\sigma) \tag{1.23}$$

we have

$$\nu_n(P_n) \xrightarrow{*} \lambda_w^\sigma. \tag{1.24}$$

In particular,

$$\nu_n(P_{n,p}^*) \xrightarrow{*} \lambda_w^\sigma. \tag{1.25}$$

1.9 Location of zeros of weighted discrete extremal polynomials.

The following theorem for an admissible triangular array is the analogue of [14, Theorem 2.2.1] and [12, Theorem 3.3.4] for sets of positive logarithmic capacity. See also [1, Remark 1.5 d].

Theorem 1.6 *Let* $0 < p \leq \infty$ *and assume the hypotheses of Theorem* 1.1. *Then the zeros of* $P_{n,p}^*$ *only accumulate in the convex hull of* S_w^* *and the number of zeros of* $P_{n,p}^*$ *lying in compact subsets of* $I \backslash S_w^*$ *is bounded uniformly in* n.

The remainder of this paper is devoted to the proofs of Theorems 1.1-1.6.

2 The Proof of Theorems 1.1 and 1.2

In this section, we proceed with the proof of Theorems 1.1 and 1.2. This will be achieved through several intermediate steps.

We first present the:

2.1 Proof of the Sufficiency of Theorem 1.1: $p = q = \infty$

Suppose first that (1.8) holds. We begin by showing that we have

$$\lim_{n\to\infty} \left(\frac{\|P_n w^n\|_{L_\infty(I)}}{\|P_n w^n\|_{L_{\infty,H}(E_n)}} \right)^{1/n} = 1. \tag{2.1}$$

We claim that we may assume without loss of generality that $P_n \in \Pi_n^*$ and has n real uniformly bounded zeros separated by the points of E_n. To see this, suppose that (2.1) holds under these hypotheses. We may further assume, using [12, Theorem 3.2.1] if necessary that I is bounded. Now let $P_n^{\#} \in \Pi_n$ satisfy

$$\frac{\|P_n^{\#} w^n\|_{L_\infty(I)}}{\|P_n^{\#} w^n\|_{L_{\infty,H}(E_n)}} = \sup\left\{\frac{\|P_n w^n\|_{L_\infty(I)}}{\|P_n w^n\|_{L_{\infty,H}(E_n)}} \mid P_n \in \Pi_n\right\}. \tag{2.2}$$

By a suitable renormalization of $P_n^{\#}$, and using [12, Theorem 3.2.1], we may assume that there exists $x_0 \in \mathrm{supp}(\mu_w)$ for which

$$|P_n^{\#} w^n|(x_0) = \|P_n^{\#} w^n\|_{L_\infty(I)} = \|P_n^{\#} w^n\|_{L_\infty(\mathrm{supp}(\mu_w))} = 1. \tag{2.3}$$

Now if $x_0 \in E_n$, then there is nothing to prove. Thus we may assume without loss of generality that $x_0 \notin E_n$ and $P_n^{\#}$ minimizes the norm

$$\|P_n w^n\|_{L_{\infty,H}(E_n)}$$

over all polynomials $P_n \in \Pi_n$ satisfying $|P_n w^n|(x_0) = 1$.

We proceed to analyze the zeros of $P_n^{\#}$ and to this end we consider an equivalent problem for monic polynomials.

We set for a given n

$$\begin{aligned} \widetilde{E}_n : &= \left\{x \in I \mid x^{-1} + x_0 \in E_n\right\}, \\ \widetilde{w}(x) : &= x^{-1} w(x^{-1} + x_0), \quad x \in \widetilde{E}_n \end{aligned}$$

and

$$Q_n^{\#}(x) := \frac{x^n P_n^{\#}(x^{-1} + x_0)}{P_n^{\#}(x_0)}, \quad x \in I.$$

Note that as I is bounded, $0 \notin \widetilde{E}_n$. Thus it is easy to see that $Q_n^{\#} \in \Pi_n^{\#}$ and minimizes $\|Q_n \widetilde{w}^n\|_{L_\infty(\widetilde{E}_n)}$ amongst all monic polynomials of precise degree n. Moreover, it is well known that $Q_n^{\#}$ has n simple zeros in the convex hull of $\widetilde{E}_n$ and its zeros are separated by the points of $\widetilde{E}_n$.

Thus $P_n^{\#}$ has at least $n-1$ real simple zeros in the convex hull of E_n separated by the points of E_n. Suppose first that $P_n^{\#}$ has degree n for every n and all its zeros are uniformly bounded. Then by a suitable renormalization of $P_n^{\#}$ and applying (2.2), we have our claim. Thus we may assume henceforth, that either $P_n^{\#}$ has degree $n-1$ with all its zeros uniformly bounded or that $P_n^{\#}$ has degree n for every n and there is one zero of $P_n^{\#}$ denoted by ε_n with

$$1/\varepsilon_n \to 0, \quad n \to \infty.$$

This is possible as recall that we assumed without loss of generality that I was bounded.

Suppose first that $P_n^{\#}$ is of degree $n-1$ with all its zeros uniformly bounded. Then we may choose a bounded sequence $\{A_n\}$ such that for each n, $|A_n| \geq$

$2\sup|I|$ and the zeros of $(x - A_n)P_n^{\#}$ are separated by the points of E_n. Now put

$$\widehat{P}_n(x) := C_n(x - A_n)P_n^*(x), \quad x \in I$$

for a suitable sequence $\{C_n\}$ chosen so that $\widehat{P}_n$ is monic for every n. Then observing that the function

$$x \longrightarrow \frac{1}{|x - A_n|}$$

is uniformly bounded on I for every n, we see that (2.1) holds for $\widehat{P}_n$ and so it holds for $P_n^{\#}$. Thus our claim again follows from (2.2).

Suppose next that $P_n^{\#}$ has degree n for every n and there is one zero ε_n of $P_n^{\#}$ with

$$1/\varepsilon_n \to 0, \quad n \to \infty.$$

Then we may define the sequences $\{A_n\}$ and $\{C_n\}$ such as before except this time we set

$$\widehat{P}_n(x) := C_n \frac{(x - A_n)}{1 - x/\varepsilon_n} P_n^*(x), \quad x \in I$$

and observe that the function

$$x \longrightarrow \frac{x - x/\varepsilon_n}{|x - A_n|}$$

is again uniformly bounded on I. This completes the proof of the claim.

We are now in a position to prove (2.1). Choose $P_n \in \Pi_n$ and without loss of generality we may assume that P_n is monic and has n simple uniformly bounded zeros separated by the points of E_n. Let $\nu_n(P_n) = \nu_n$ be the normalized zero counting measure of P_n. As E_n is admissible, we may assume, (by taking subsequences if necessary), that the measures ν_n converge weak* to a probability measure ν where ν has compact support in I and $\nu \le \sigma$. Now we write

$$\|P_n w^n\|_{L_\infty(I)} \le \|P_n \exp(nU^\nu)\|_{L_\infty(I)} \|\exp(-nU^\nu) w^n\|_{L_\infty(I)}$$

and observe first that the weak* convergence above, [12, Theorem 3.2.1] and the continuity of U^ν guarantee that

$$\limsup_{n\to\infty} \|P_n \exp(nU^\nu)\|_{L_\infty(I)}^{1/n} = 1.$$

Thus to prove (2.1), it is enough to show that

$$\liminf_{n\to\infty} \|P_n w^n\|_{L_{\infty,H}(E_n)}^{1/n} \ge \|\exp(-U^\nu) w\|_{L_\infty(I)}.$$

We break down the proof into several steps.

Step I: We first show that given any $\zeta > 0$, there exists a point $y_0 \in \mathrm{supp}(\sigma - \nu)$ satisfying (1.4) for which

$$\big|\exp(-U^\nu)w\big|(y_0) > \|\exp(-U^\nu)w\|_{L_\infty(I)} - \zeta.$$

We first claim that $U^{\mu_w - \nu}$ is subharmonic in

$$\overline{\mathbb{C}}\backslash\big(\mathrm{supp}(\sigma-\nu)\cap\mathrm{supp}(\mu_w)\big).$$

To see this, observe first that $\mu_w \le \nu$ outside $\mathrm{supp}(\sigma-\nu)\cap\mathrm{supp}(\mu_w)$. Thus the positive part of the signed measure $\mu_w - \nu$ is supported in $\mathrm{supp}(\sigma-\nu)\cap\mathrm{supp}(\mu_w)$ and thus gives rise to a subharmonic function in $\overline{\mathbb{C}}\backslash\big(\mathrm{supp}\big(\sigma-\nu\big)\cap\mathrm{supp}(\mu_w)\big)$. The negative part of the measure on the other hand always gives rise to a subharmonic function in $\overline{\mathbb{C}}$, see ([12], Chapter 0, Theorem 5.6). Thus we have our claim. Now the maximum principle for subharmonic functions implies that $U^{\mu_w-\nu}$ attains its maximum on $\mathrm{supp}(\sigma-\nu)\cap\mathrm{supp}(\mu_w)$. Recalling that

$$\log w(x) - U^\nu(x)\begin{cases} = U^{\mu_w-\nu}(x) - F_w, & x \in \mathrm{supp}(\mu_w) \\ \le U^{\mu_w-\nu}(x) - F_w, & x \in I \end{cases}$$

immediately shows that $w\exp(-U^\nu)$ attains its maximum on $\mathrm{supp}(\sigma-\nu)\cap\mathrm{supp}(\mu_w)$ and so there exists $y_0^* \in \mathrm{supp}(\sigma-\nu)\cap\mathrm{supp}(\mu_w)$ for which

$$w(y_0^*)\exp\big(-U^\nu(y_0^*)\big) = \|\exp(-U^\nu)w\|_{L_\infty(I)}.$$

Now we use the continuity of $w\exp(-U^\nu)$ to deduce that there exists a neighborhood V of y_0^* with

$$\mathrm{cap}\big(\mathrm{supp}(\sigma-\nu)\cap V\big) > 0$$

and such that for all $y \in V$

$$\big|\exp(-U^\nu)w\big|(y) > \|\exp(-U^\nu)w\|_{L_\infty(I)} - \zeta.$$

Finally, recalling that (1.4) holds q.e., we apply the identity, (see [6], Theorem 2.6)

$$\mathrm{supp}(\mu_w) \subseteq \mathrm{supp}(\lambda_w^\sigma)$$

and choose y_0 to satisfy (1.4) as well.

Step II: For the given point y_0, we now establish the identity

$$\liminf_{n\to\infty}\|P_n w^n\|^{1/n}_{L_{\infty,H}(E_n)} \ge w(y_0)\exp(-U^\nu\big(y_0)\big).$$

Then combining the above equation with the argument above and letting $\zeta \to 0$ establishes (2.1).

To this end, for a given sufficiently large $n \ge n_0$, put $\Delta_{1/n} := (y_0 - 1/n, y_0 + 1/n)$. We may choose n so large that $|\Delta_{1/n}| < 1/2$. Now choose $0 < \delta < 1/n$ and similarly define Δ_δ. We write $P_n = T_n S_n$ where T_n is a monic polynomial whose zeros coincide with those of P_n in $\Delta_{1/n}$ and S_n is a monic polynomial whose zeros coincide with the zeros of P_n in $I\backslash\Delta_{1/n}$.

First let ν_1 denote the restriction of the measure ν to $I\backslash\Delta_{1/n}$. Then applying the continuity of U^{ν_1} and the weak* convergence of ν_n yields

$$\lim_{n\to\infty} |S_n|^{1/n}(x) \geq \exp\big(-U^{\nu}(y_0)\big).$$

We now estimate the polynomial T_n.

Recall first that $y_0 \in \operatorname{supp}(\sigma-\nu)$. Let l_n denote the number of zeros of T_n in Δ_δ and m_n the number of points in $E_n \cap \Delta_\delta$. It follows that as n is sufficiently large, $\nu_n(T_n)(\Delta_\delta) < \sigma_n(\Delta_\delta)$ and thus m_n is much larger than l_n. Since the intervals

$$\left(\frac{\eta_{\pm i-1,n}+\eta_{\pm i,n}}{2}, \frac{\eta_{\pm i,n}+\eta_{\pm i+1,n}}{2}\right),\ i=0,1,2,...$$

contain exactly one point of E_n, there exists at least one such interval which contains no zeros of P_n and whose centre is in Δ_δ. Let us denote this centre by $\eta_{j,n}$ and its adjacent points by $\eta_{j-1,n}$ and $\eta_{j+1,n}$ respectively. Recalling that the zeros of P_n separate the points of E_n and using the fact that $|\Delta_{1/n}| < 1/2$ yields the following estimate on T_n:

$$\begin{aligned}
|T_n(\eta_{j,n})|^{1/n} &\geq \left(\frac{|\eta_{j,n}-\eta_{j-1,n}||\eta_{j,n}-\eta_{j+1,n}|}{4}\right)^{1/n} \times \\
&\times \left(\prod_{\substack{\eta_{\pm i,n}\in\Delta_{1/n}\\ \eta_{\pm i,n}\neq\eta_{j,n}}} |\eta_{j,n}-\eta_{\pm i,n}|\right)^{1/n} \\
&\geq (1/4)^{1/n}\left(\prod_{\substack{\eta_{\pm i,n}\in\Delta_{1/n}\\ \eta_{\pm i,n}\neq\eta_{j,n}}} |\eta_{j,n}-\eta_{\pm i,n}|\right)^{2/n}.
\end{aligned}$$

Observe that

$$\eta_{j,m} \to y_0,\ m\to\infty.$$

Thus applying (1.4)and the dominated convergence theorem gives

$$\liminf_{n\to\infty} |T_n(\eta_{j,n})|^{1/n} \geq 1.$$

Combining our arguments above yields

$$\liminf_{n\to\infty} \|P_n w^n\|^{1/n}_{L_{\infty,H}(E_n)} \geq \exp\big(-U^{\nu}(y_0)\big)w(y_0)$$

as required. This completes the proof of Theorem 1.1 for $p=q=\infty$. We provide the remaining details for the proof of Theorem 1.1 later in Section 2.4. $\square$

2.2 The Proof of Theorem 1.2

We now present:

Proof of Theorem 1.2 For the given weight w, let us recall, [12, Theorem 1.1.3], that there exists a unique constant F_w such that

$$\begin{cases} U^{\mu_w}(x) - \log w(x) = F_w, & x \in \mathrm{supp}(\mu_w) \\ U^{\mu_w}(x) - \log w(x) \geq F_w, & x \in I. \end{cases}$$

We set:

$$w_0(x) := \min\big(w(x), \exp(U^{\lambda_w^\sigma}(x) - F_w^\sigma)\big), \quad x \in I. \tag{2.4}$$

It is straightforward to check that w_0 satisfies both (1.1) and (1.2). Moreover, the uniqueness of the equilibrium measure μ_w and F_w^σ together with (1.11) and (1.12) easily give that:

(a)

$$w_0(x) := \begin{cases} \exp(U^{\lambda_w^\sigma}(x) - F_w^\sigma), & x \in \mathrm{supp}(\lambda_w^\sigma) \\ w(x), & \text{otherwise} \end{cases}. \tag{2.5}$$

(b)

$$\begin{aligned} w_0(x) &= w(x), \quad x \in \overline{I \backslash S_0} \\ w_0(x) &\leq w(x), \quad x \in I. \end{aligned} \tag{2.6}$$

(c)

$$\mu_{w_0} = \lambda_w^\sigma. \tag{2.7}$$

(d)

$$F_{w_0} = F_w^\sigma. \tag{2.8}$$

We claim that (1.7) holds with $p = q = \infty$ with w_0. Firstly (1.3) and (2.7) show that (1.8) holds with w_0. Thus it remains to show (1.4). To see this, let I_0 be a bounded interval with $\mathrm{supp}(\lambda_{w_0}^\sigma) \subseteq I_0 \subset I$. Set

$$R_n(x) := \prod_{\eta_{\pm i,n} \in I_0} (x - \eta_{\pm i,n}),\ x \in I$$

and let $\sigma_1 := \sigma|_{I_0}$. Then (1.4) follows from (2.7) and the identity, (see [6], Theorem 2.6),

$$\mathrm{supp}(\mu_w) \subseteq \mathrm{supp}(\lambda_w^\sigma)$$

since

$$\mathrm{supp}(\lambda_w^\sigma) = \mathrm{supp}(\mu_{w_0}) \subseteq \mathrm{supp}(\lambda_{w_0}^\sigma) \subseteq I_0.$$

Thus we may apply (1.7) with w_0 together with (2.5), (2.6) and [12, Theorem 3.2.1] to obtain (1.13). □

2.3 Nikolśki Inequality

In this section, we prove a Nikolśki inequality which in this sharp form is new even under the weaker conditions of [9, Lemma 8.3(b)]. We have:

Theorem 2.1-Nikolśkii Inequality *Assume the hypotheses of Theorem* 1.1.

(a) *Then uniformly for any polynomial $P_n \in \Pi_n$ and $n \geq 1$,*

$$1 \leq \left(\frac{\|P_n w^n\|_{L_{p,H}(E_n)}}{\|P_n w^n\|_{L_{\infty,H}(E_n)}} \right) \leq C n^{1/p},\ 0 < p < \infty. \tag{2.9}$$

(b) *Moreover, there exists $\delta > 0$ such that uniformly for every polynomial $P_n \in \Pi_n$ and $n \geq 1$*

$$1 - \exp(-\delta n) \leq \left(\frac{\|P_n w^n\|_{L_{p,H}(E_n)}}{\|P_n w^n\|_{L_{q,H}(E_n)}} \right) \leq C n^{1/p-1/q},\ 0 < p < q < \infty. \tag{2.10}$$

Proof We define for $k > 0$, the compact set S_k given by (1.10) and set for $n \geq 1$:

$$E_{n,k} := S_k \cap E_n.$$

We will write

$$\mathrm{card}(E_{n,k})$$

to mean the cardinality of the set $E_{n,k}$ which is well defined.

Let $\varepsilon > 0$. Observe first that there exist positive numbers δ and δ_1 so that for all $x \in I \backslash S_\varepsilon$,

$$\begin{aligned} & U^{\lambda_w^\sigma}(x) + Q(x) - F_w^\sigma - \varepsilon \\ & \geq Q(x) - (1+\delta_1)\log|x| + \delta. \end{aligned} \tag{2.11}$$

Applying (1.13) with (2.11), we deduce that

$$\begin{aligned} & ||P_n w^n||^p_{L_{p,H}(E_n \backslash S_\varepsilon)} \leq \\ & \leq ||P_n w^n||^p_{L_{p,H}(E_n)} \exp(-\delta n p) \sum_{x \in E_n} |x^{1+\delta_1} w(x)|^{np} \\ & \leq ||P_n w^n||^p_{L_{p,H}(E_n)} \exp(-\delta n p) ||(x^{1+\delta_1} w(x))^n||^p_{L_{p,H}(E_n)}. \end{aligned} \tag{2.12}$$

Now let us apply (1.5) to (2.12). We have shown the following: For every $\varepsilon > 0$, there exists a $\delta > 0$ and N_0 such that for $n \geq N_0$ and for every polynomial $P_n \in \Pi_n$

$$\|P_n w^n\|_{L_{p,H}(E_n \backslash S_\varepsilon)} \leq \|P_n w^n\|_{L_{p,H}(E_n)} \exp(-\delta n).$$

From the above, we conclude that for every $\varepsilon > 0$, there exists a $\delta > 0$ and N_0 such that for $n \geq N_0$ and for every polynomial $P_n \in \Pi_n$

$$1 \leq \left(\frac{\|P_n w^n\|_{L_{p,H}(E_n)}}{\|P_n w^n\|_{L_{p,H}(E_{n,\varepsilon})}} \right) \leq \frac{1}{1 - \exp(-\delta n)}. \qquad (2.13)$$

Now let $0 < p < q \leq \infty$ and $\varepsilon > 0$. We claim that for every polynomial $P_n \in \Pi_n$, $n \geq 1$,

$$\min \left\{ 1, \mathrm{card}(E_{n,\varepsilon})^{1/q-1/p} \right\} \leq \frac{\|P_n w^n\|_{L_{p,H}(E_{n,\varepsilon})}}{\|P_n w^n\|_{L_{q,H}(E_{n,\varepsilon})}} \leq \mathrm{card}(E_{n,\varepsilon})^{1/p-1/q}. \qquad (2.14)$$

To see this, observe first that (2.14) follows for $0 < p < q < \infty$ from Hőlders inequality. For $q = \infty$, it again persists by definition of the discrete Holder norm (for the lower bound) and by a trivial estimation (for the upper bound).

To complete the proof of Theorem 2.1, we need to invoke (2.13), (2.14) and the fact that

$$(\mathrm{card}(E_{n,\varepsilon}))^{1/p-1/q} = O\left(n^{1/p-1/q} \right)$$

which we may obtain using distribution condition A. Observe that if $q \neq \infty$, then Theorem 2.1 follows easily. If $q = \infty$, the right most inequality in (2.9) follows using the same argument as above and the fact that

$$E_{n,\varepsilon} \subseteq E_n$$

while the left most inequality in (2.9) is true by inspection. We have proved Theorem 2.1. □

2.4 The Proof of Theorem 1.1

We now provide the remaining details in the

The Proof of Theorem 1.1 We claim that (2.1) implies

$$\lim_{n \to \infty} \left(\frac{\|P_n w^n\|_{L_q(I)}}{\|P_n w^n\|_{L_{p,H}(E_n)}} \right)^{1/n} = 1 \qquad (2.15)$$

as required. To see this, we observe first using (2.9)and (2.10), that it is immediate that

$$\lim_{n \to \infty} \left(\frac{\|P_n w^n\|_{L_{p,H}(E_n)}}{\|P_n w^n\|_{L_{q,H}(E_n)}} \right)^{1/n} = 1.$$

Next, we recall first that by [12, Theorem 3.6.2], there exists a compact set $B \subset I$ such that

$$\lim_{n \to \infty} \left(\frac{\|P_n w^n\|_{L_p(I)}}{\|P_n w^n\|_{L_p(B)}} \right)^{1/n} = 1.$$

Thus we may cover B by a bounded interval $J \subset I$ and obtain using the above and [14, Lemma 2.1.7] that

$$\begin{aligned}\limsup_{n\to\infty} \|P_n w^n\|_{L_p(I)}^{1/n} &\leq \limsup_{n\to\infty} \|P_n w^n\|_{L_p(B)}^{1/n} \\ &\leq \limsup_{n\to\infty} \|P_n w^n\|_{L_\infty(J)}^{1/n} \\ &\leq \limsup_{n\to\infty} \|P_n w^n\|_{L_\infty(I)}^{1/n}.\end{aligned}$$

Similarly using [12, Theorem 3.2.1], we can easily show that

$$\liminf_{n\to\infty} \|P_n w^n\|_{L_p(I)}^{1/n} \geq \liminf_{n\to\infty} \|P_n w^n\|_{L_\infty(I)}^{1/n}.$$

(2.15) then follows easily and we have shown the sufficiency of (1.8).

Next suppose (1.7) holds. Then using a similar argument to the above, we may conclude that for every sequence of polynomials $P_n \in \Pi_n$

$$\lim_{n\to\infty} \left(\frac{\|P_n w^n\|_{L_\infty(I)}}{\|P_n w^n\|_{L_{\infty,H}(E_n)}} \right)^{1/n} = 1. \tag{2.16}$$

Assume that (1.8) does not hold. Firstly, we recall that that there exists a unique constant F_w satisfying the variational conditions:

$$\begin{cases} U^{\mu_w}(x) - \log w(x) = F_w, & x \in \text{supp}(\mu_w) \\ U^{\mu_w}(x) - \log w(x) \geq F_w, & x \in I. \end{cases}$$

We claim that

$$F_w < F_w^\sigma. \tag{2.17}$$

To see this, we first observe that (1.11), (1.12) and the variational conditions above give

$$U^{\lambda_w^\sigma}(x) - U^{\mu_w}(x) \leq F_w^\sigma - F_w,\ x \in \text{supp}(\lambda_w^\sigma).$$

By the principle of Domination ([12], Theorem 3.3.1), we infer that

$$U^{\lambda_w^\sigma}(x) - U^{\mu_w}(x) \leq F_w^\sigma - F_w,\ x \in \mathbb{C}. \tag{2.18}$$

Letting $|x| \to \infty$ in (2.18) we learn that

$$F_w^\sigma \geq F_w.$$

But by assumption, (1.8) does not hold. Thus as $\lambda_w^\sigma \leq \sigma$ it follows that

$$\lambda_w^\sigma \neq \mu_w.$$

As the measures λ_w^σ and μ_w are both supported on the real line, (2.17) follows. Next, using (4.3) below (which is independent of the proof of Theorem 1.1), we know that there exists a sequence of polynomials $Q_n \in \Pi_n^*$ satisfying

$$\limsup_{n\to\infty} \|Q_n w^n\|_{L_{\infty,H}(E_n)}^{1/n} \leq \exp(-F_w^\sigma).$$

Thus using (2.17), we have

$$\limsup_{n\to\infty} \|Q_n w^n\|_{L_{\infty,H}(E_n)}^{1/n} < \exp(-F_w). \tag{2.19}$$

Then (2.19) and the identity, see [12, Theorem 3.2.1],

$$\|Q_n w^n\|_{L_\infty(I)} = \|Q_n w^n\|_{L_\infty(\mathrm{supp}(\mu_w))} \geq \exp(-nF_w)$$

give

$$\liminf_{n\to\infty} \left(\frac{\|Q_n w^n\|_{L_\infty(I)}}{\|Q_n w^n\|_{L_{\infty,H}(E_n)}} \right)^{1/n} > 1.$$

This last equation contradicts (2.16) and so (1.8) must hold. This completes the proof of Theorem 1.1. □

3 The Proofs of Theorems 1.5 and 1.6

We begin with the

Proof of Theorem 1.5 Firstly define w_0 as in (2.4). Then given $P_n \in \Pi_n^*$ we always have, using (2.6), the relation

$$\liminf_{n\to\infty} \|P_n w^n\|_{L_{\infty,H}(E_n)}^{1/n} \geq \liminf_{n\to\infty} \|P_n w_0^n\|_{L_{\infty,H}(E_n)}^{1/n}. \tag{3.1}$$

Thus using (2.1), (2.8) and the identity, see [12, Theorem 3.4.1],

$$\|P_n w^n\|_{L_\infty(I)} = \|P_n w^n\|_{L_\infty(\mathrm{supp}(\mu_w))} \geq \exp(-nF_w)$$

we may write (3.1) as

$$\begin{aligned} \liminf_{n\to\infty} \|P_n w^n\|_{L_{\infty,H}(E_n)}^{1/n} &\geq \liminf_{n\to\infty} \|P_n w_0^n\|_{L_\infty(I)}^{1/n} \\ &\geq \exp(-F_{w_0}) = \exp(-F_w^\sigma). \end{aligned} \tag{3.2}$$

This last inequality establishes (1.21) for $p = \infty$ and the lower bound in (1.22) for $p = \infty$. We now claim the existence of a sequence of monic polynomials $Q_n \in \Pi_n^*$ satisfying

$$\limsup_{n\to\infty} \|Q_n w^n\|_{L_{\infty,H}(E_n)}^{1/n} \leq \exp(-F_w^\sigma). \tag{3.3}$$

Notice that if we can establish (3.3), then the minimality of $P_{n,\infty}^*$ yields

$$\limsup_{n\to\infty} \|P_{n,\infty}^* w^n\|_{L_{\infty,H}(E_n)}^{1/n} \leq \exp(-F_w^\sigma). \tag{3.4}$$

(3.4) together with (1.21) will then imply (1.22) for $p = \infty$. Moreover if $p \neq \infty$, Theorem 2.1 and the minimality of $P_{n,p}^*$ yields

$$\begin{aligned} &\limsup_{n\to\infty} \|P_{n,p}^* w^n\|_{L_{p,H}(E_n)}^{1/n} \\ &\leq \limsup_{n\to\infty} \|Q_n w^n\|_{L_{p,H}(E_n)}^{1/n} \\ &\leq \limsup_{n\to\infty} \|Q_n w^n\|_{L_{\infty,H}(E_n)}^{1/n} \\ &\leq \exp(-F_w^\sigma) \end{aligned}$$

so that again we have (1.22). Also, if (1.23) holds, then we may apply it with the weight w_0 given by (2.4). If we do this and also apply Theorems 1.1 and 2.1 recalling that (1.8) is now satisfied, we obtain that

$$\lim_{n\to\infty} \|P_n w_0^n\|_{L_\infty(I)}^{1/n} = \exp(-F_{w_0}).$$

But then by [12, Theorem 3.4.1], we must have

$$\nu(P_n) \xrightarrow{*} \mu_{w_0} = \lambda_w^\sigma.$$

(1.25) will then follow from (1.24) using (1.22). Thus everything boils down to proving (3.3).

Our method of proof makes use of Theorem 1.1, the weight w_0 defined in (2.4) and a delicate construction of the polynomials in question by specifying their zeros and carefully discretizing the measure λ_w^σ. Some of our ideas appeared first in ([10], Lemma 4.1) but we provide full details for the readers convenience.

We choose $\varepsilon > 0$ small enough, set

$$S_{-\epsilon} := \left\{x \in I \mid U^{\lambda_w^\sigma} + Q(x) \geq F_w^\sigma - \epsilon\right\}$$

and consider

$$\overline{I\backslash S_{-\varepsilon}} =: \left\{x \in I \mid U^{\lambda_w^\sigma} + Q(x) \leq F_w^\sigma - \epsilon\right\}.$$

We now break up our argument into several steps.

Step I: We first show that for sufficiently large n,

$$\sigma_n(\overline{I\backslash S_{-\varepsilon}}) < 1.$$

Observe that $\overline{I\backslash S_{-\varepsilon}}$ is compact with $\partial(\overline{I\backslash S_{-\varepsilon}}) = 0$. Here ∂ denotes the usual topological boundary. This is possible by the continuity of $U^{\lambda_w^\sigma}$ and Q and by choosing ϵ small enough. Moreover (1.11) implies that $\overline{I\backslash S_{-\varepsilon}} \subset \mathrm{supp}(\lambda_w^\sigma)$ and so consequently,

$$\lambda_w^\sigma(\overline{I\backslash S_{-\varepsilon}}) < 1. \tag{3.5}$$

Next we observe that Condition A implies that the measure σ has no mass points. Thus, see ([2], Theorem 25.8), we have

$$\lim_{n\to\infty} \sigma_n(K) = \sigma(K)$$

for every compact $K \subset I$ with $\sigma(\partial K) = 0$. In particular, applying the above with $K = \overline{I\backslash S_{-\varepsilon}}$ gives

$$\lim_{n\to\infty} \sigma_n(\overline{I\backslash S_{-\varepsilon}}) = \sigma(\overline{I\backslash S_{-\varepsilon}}). \tag{3.6}$$

Now observe that (1.11) and (1.12) imply easily that $\sigma = \lambda_w^\sigma$ on $\overline{I\backslash S_{-\varepsilon}}$ and thus (3.5) together with (3.6) imply that for sufficiently large n

$$\sigma_n(\overline{I\backslash S_{-\varepsilon}}) < 1. \tag{3.7}$$

This completes the proof of Step 1.

Step II: We construct monic polynomials P_n with n zeros for which:

(a)

$$E_n \cap \overline{I\backslash S_{-\varepsilon}} \subset Z(P_n) \subset \mathrm{supp}(\lambda_w^\sigma) \tag{3.8}$$

and

(b)

$$n\big(\nu_n(P_n)\big) \xrightarrow{*} \lambda_w^\sigma, \quad n \to \infty. \tag{3.9}$$

Here, $Z(P_n)$ denotes the zero set of P_n.

To do this, we proceed as follows. Choose $n_1 := n\big(1 - \sigma_n(\overline{I\backslash S_{-\varepsilon}})\big)$ zeros of P_n in $\mathrm{supp}(\lambda_w^\sigma)\backslash(\overline{I\backslash S_{-\varepsilon}})$ which we denote by x_{i,n_1}, $1 \le i \le n_1$ and satisfying

$$\lambda_w^\sigma\big([x_{i,n_1}, x_{i+1,n_1}]\big) = 1/n, \quad 1 \le i \le n_1.$$

Now as λ_w^σ has no mass points, for any fixed $a \in I$, the function $\lambda_w^\sigma\big([a,x]\big)$ is a continuous function of x and so

$$\nu_n(P_n)|_{\mathrm{supp}(\lambda_w^\sigma)\backslash(\overline{I\backslash S_{-\varepsilon}})} \xrightarrow{*} \lambda_w^\sigma|_{\mathrm{supp}(\lambda_w^\sigma)\backslash(\overline{I\backslash S_{-\varepsilon}})}. \tag{3.10}$$

The remaining $n\sigma_n(\overline{I\backslash S_{-\varepsilon}}) < n$ zeros of P_n we take from the set $E_n \cap \overline{I\backslash S_{-\varepsilon}}$. Then finally recalling that $\sigma = \lambda_w^\sigma$ on $\overline{I\backslash S_{-\varepsilon}}$ and using (3.10) yields (3.8) and (3.9).

Step III: Completion of the proof of (3.3).

First note that $P_n = 0$ on $E_n \cap \overline{I\backslash S_{-\varepsilon}}$. Thus using the definition of $\overline{I\backslash S_{-\varepsilon}}$ and (3.8), we must have

$$\|P_n w^n\|_{L_{\infty,H}(E_n)} \le \exp(-F_w^\sigma + \epsilon)n \|P_n \exp(nU^{\lambda_w^\sigma})\|_{L_\infty(\mathrm{supp}(\lambda_w^\sigma))}. \tag{3.11}$$

Moreover, by (3.8), (3.9), (2.7) and [12, Theorem 3.4.1] we have

$$\lim_{n\to\infty} \|P_n w_0^n\|_{L_\infty(\mathrm{supp}(\mu_{w_0}))}^{1/n} = 1$$

where w_0 was defined by (2.4). But then using (2.5) this implies that

$$\lim_{n\to\infty} \left\|P_n \exp(nU^{\lambda_w^\sigma})\right\|_{L_\infty(\mathrm{supp}(\lambda_w^\sigma))}^{1/n} = 1. \tag{3.12}$$

Substituting (3.12) into (3.11) and letting $\varepsilon \to 0+$ gives (3.3). $\square$

The Proof of Theorem 1.6 This follows using (1.13), [14, Lemma 1.3.2] and [12, Theorem 3.3.4]. $\square$

4 The Proofs of Theorems 1.3 and 1.4

In this section, we present the proofs of Theorem's 1.3 and 1.4. We begin with the

Proof of Theorem 1.3 We first claim that the following holds:

$$\lim_{n\to\infty} |P_n^* w^n|^{1/n}(x) = \exp\big(- U^{\lambda_w^\sigma}(x) - Q(x)\big) \tag{4.1}$$

for *q.e.* $x \in I$.

To see this, we first observe that λ_w^σ and the measures $\{\nu_n(P_n^*)\}$, $n = 1, 2, ...$ are of compact support on I. Thus we may invoke the Lower Envelope theorem, see ([12], Chapter 1, Theorem 6.9), to deduce that

$$\lim_{n\to\infty} U^{\nu_n(P_n^*)}(x) = U^{\lambda_w^\sigma}(x) \tag{4.2}$$

for *q.e.* $x \in I$. Letting $\zeta_{k,n}$, $1 \le k \le n$ denote the zeros of P_n^*, we may write

$$\begin{aligned} -(1/n)\log|P_n^*|(x) &= 1/n \sum_{k=1}^{n} \log\frac{1}{|x-\zeta_{k,n}|} \\ &= \int \log\frac{1}{|x-t|} d\nu_n(P_n^*) = U^{\nu_n(P_n^*)}(x) \end{aligned}$$

and then easily deduce (4.1) from (4.2). We proceed by contradiction. Suppose that

$$\mathrm{cap}(A\backslash(\overline{I\backslash S_0})) \neq 0.$$

Fix $y \in A\backslash(\overline{I\backslash S_0})$ so that (4.1) holds. Then by the definition of the set $\overline{I\backslash S_0}$ we must have

$$-U^{\lambda_w^\sigma}(y) - Q(y) > -F_w^\sigma. \tag{4.3}$$

Combining (4.1) with (4.3) then implies that

$$\liminf_{n\to\infty} |P_n^* w^n|^{1/n}(y) = \exp\big(- U^{\lambda_w^\sigma}(y) - Q(y)\big)$$

$$> \exp(-F_w^\sigma). \tag{4.4}$$

Thus (4.4) and (1.22) imply that

$$\liminf_{n\to\infty} \left(\frac{\|P_n^* w^n\|_{L_\infty(A)}}{\|P_n^* w^n\|_{L_{\infty,H}(E_n)}} \right)^{1/n} \geq \liminf_{n\to\infty} \left(\frac{\|P_n^* w^n\|_{L_\infty(A\backslash(\overline{I\backslash S_0}))}}{\|P_n^* w^n\|_{L_{\infty,H}(E_n)}} \right)^{1/n}$$

$$\geq \liminf_{n\to\infty} \left(\frac{|P_n^* w^n|(y)}{\|P_n^* w^n\|_{L_{\infty,H}(E_n)}} \right)^{1/n}$$

$$> \exp(-F_w^\sigma + F_w^\sigma) = 1.$$

This last statement contradicts (1.16) and so we have completed the proof of the theorem. □

We now proceed with the

Proof of Theorem 1.4 Firstly, as S is compact, (1.18) follows immediately using Theorem 1.3 and Theorem 2.1. To see (1.19), we may assume firstly because of Theorem 2.1 that $p = q$. Next we observe that (1.11) and (1.12) imply that

$$U^{\lambda_w^\sigma}(x) + Q(x) = F_w^\sigma \tag{4.5}$$

for every $x \in S$. Applying the method of Theorem 1.3 above, we may fix $y \in S$ such that

$$\lim_{n\to\infty} |P_{n,p}^* w^n|^{1/n}(y) = \exp(-F_w^\sigma). \tag{4.6}$$

Then (1.22) and (4.6) easily yield,

$$\liminf_{n\to\infty} \left(\frac{\|P_{n,p}^* w^n\|_{L_\infty(S)}}{\|P_{n,p}^* w^n\|_{L_{p,H}(E_n)}} \right)^{1/n} \tag{4.7}$$

$$\geq \liminf_{n\to\infty} \left(\frac{|P_{n,p}^* w^n|(y)}{\|P_{n,p}^* w^n\|_{L_{p,H}(E_n)}} \right)^{1/n}$$

$$\geq \exp(-F_w^\sigma + F_w^\sigma) = 1.$$

Now we apply the method of [14, Lemma 2.1.7] and the above to deduce that

$$\liminf_{n\to\infty}\left(\frac{\|P_{n,p}^{*}w^{n}\|_{L_p(S)}}{\|P_{n,p}^{*}w^{n}\|_{L_{p,H}(E_n)}}\right)^{1/n} \geq \liminf_{n\to\infty}\left(\frac{\|P_{n,p}^{*}w^{n}\|_{L_\infty(S)}}{\|P_{n,p}^{*}w^{n}\|_{L_{p,H}(E_n)}}\right)^{1/n} \geq 1. \quad (4.8)$$

This last inequality establishes (1.19). We note that (4.7) holds if S has positive logarithmic capacity and that we only require S to be a finite union of finite non degenerate intervals in the transition from L_p to L_∞ in (4.8). Finally to see (1.20), we first recall, see (2.4) above, that there exists a continuous, not identically zero weight

$$w_0 : I \to [0, \infty)$$

satisfying (1.2) and the following:

$$\begin{aligned} w_0(x) &= w(x), \quad x \in S_0. \\ w_0(x) &\leq w(x), \quad x \in I. \\ \mu_{w_0} &= \lambda_w^{\sigma}. \\ F_{w_0} &= F_w^{\sigma}. \end{aligned}$$

Indeed, w_0 is given by the formula

$$w_0(x) := \min\left\{w(x), \exp\left(U^{\lambda_w^{\sigma}}(x) - F_w^{\sigma}\right)\right\}, \, x \in I.$$

Observe first that $w = w_0$ on S. Define:

$$S_w^{**} := \left\{x \in I : U^{\mu_w}(x) + Q(x) \leq F_w\right\}.$$

Then using the definition of w_0, we observe that if N is a given neighborhood of S_0, then N is the same neighborhood for $S_{w_0}^{**}$. Thus we may apply, [12, Theorem 3.6.1] to deduce that

$$\begin{aligned} \liminf_{n\to\infty}\|P_{n,p}^{*}w^{n}\|_{L_p(N)}^{1/n} &\geq \liminf_{n\to\infty}\|P_{n,p}^{*}w_0^{n}\|_{L_p(N)}^{1/n} \\ &\geq \liminf_{n\to\infty}\|P_{n,p}^{*}w_0^{n}\|_{L_p(I)}^{1/n} \\ &\geq \liminf_{n\to\infty}\|P_{n,p}^{*}w_0^{n}\|_{L_\infty(I)}^{1/n} \\ &\geq \liminf_{n\to\infty}\|P_{n,p}^{*}w^{n}\|_{L_\infty(S)}^{1/n}. \end{aligned}$$

Recalling that (4.7) holds if S has positive logarithmic capacity and applying the above inequality yields the result. □

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