On Golden Gates and Discrepancy

Examining the Efficiency of Universal Gate Sets

Brent Mode

University of Louisville

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Advisor: Dr. Steven Damelin

UNIVERSITY OF MICHIGAN

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- All classical programs are formed from a combination of AND, OR, and NOT gates.
- These programs are synthesized exactly, since the spectrum of possible programs is discrete.
- In other words, if you can dream it, it can be done exactly.

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- A 1-qubit quantum gate X acts on $|\psi\rangle$ to produce $|\psi'\rangle$.
- While classical logic gates are discrete, X can be any 2×2 matrix such that, since $|\psi|^2 = 1$, then $|\psi'|^2 = 1$.

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- *Projective Special Unitary Group* - Further, for quantum gates it is also valid to view the gates X and $-X$ as the same, which leads us to: $PSU(2) = SU(2)/Z(SU(2))$.
- *Metric on* $SU(2)$ - We need to define a notion of distance on $SU(2)$, so we use the invariant metric, $d^2_{SU(2)}(X, Y) = 1 - \dfrac{Tr|X^\dagger Y|}{2}$, where $d : SU(2) \to \mathbb{R}_{\geqslant 0}$.

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- Unlike in classical computing, it is impossible to exactly synthesize every possible program using a handful of gates.
- This is the same problem that occurs when comparing the rational numbers to the real numbers.
- What is needed is a way to approximate every element of $SU(2)$ using a circuit built from a small set of specially chosen quantum gates.
- The problem is then two-fold: Find a good gate set and come up with an approximation algorithm.

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- These elements are combined to form reduced words of increasing length, with iX, iY, and iZ then inserted at the front to quadruple the number of elements of a certain length.
- We say that $\Omega = \langle T \rangle$ is the group generated by T.
- Then $V(t)$ is defined as the set of elements in Ω of length at most t.

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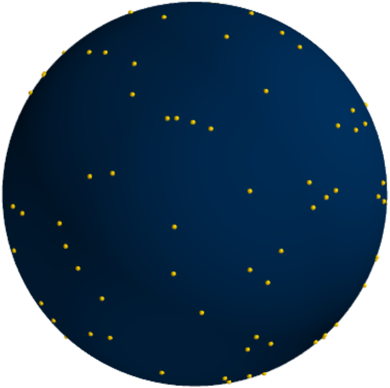
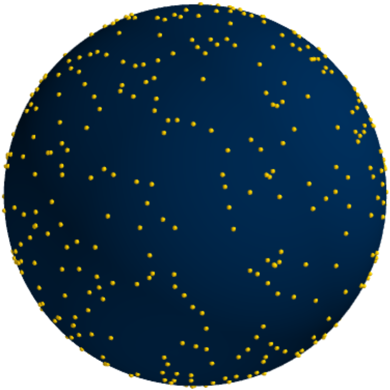
- This is a well-studied problem in number theory and lends itself to being studied numerically.
- In many ways, we can change the quantum problem to a study of how well this point set is distributed on the sphere.

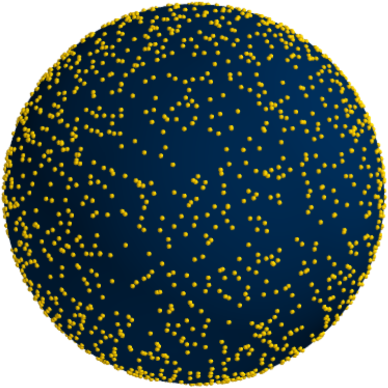
The Points of $V(2)$



The Points of $V(3)$



The Points of $V(4)$



Efficiency and Discrepancy

Solovay-Kitaev and Efficiency

- The Solovay-Kitaev Theorem states that for $X \in SU(2)$ and a symmetric universal set of quantum gates, for a given $\varepsilon > 0$, there exists some $\omega \in \Omega$ of length $O\left(\log^c\left(\frac{1}{\varepsilon}\right)\right)$ approximating X within distance ε.

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- This guarantees that an approximation exists, but does not robustly address the relative efficiency of different choices of gate set.
- To that end, Sarnak introduces the covering exponent, defined below, to serve this purpose:

$$K(T) \equiv \limsup_{\varepsilon \to 0} \frac{\log|V(t_\varepsilon)|}{\log(\frac{1}{\mu(B(\varepsilon))})},$$

where t_ε is the smallest t such that for the given ε, $V(t_\varepsilon)$ approximates all of $SU(2)$ within a distance ε, $B(\varepsilon)$ is an arbitrary ball of radius ε in $SU(2)$ and μ is a Haar measure on $SU(2)$.

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- This is not the case: Sarnak has proven that $\frac{4}{3} \leqslant K(T) \leqslant 2$.
- However, T is optimally efficiency *almost everywhere*.
- It is suspected that $K(T) = \frac{4}{3}$; what remains is for this to be proven or refuted.

Conjecture on K

- We conjecture $\varepsilon \leq f(t_\varepsilon)5^{-t_\varepsilon/4}$ for a function $f:(0,\infty)\to(1,\infty)$ satisfying:

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- Then $M \sim f(\log N)N^{-1/4}$. Here $N \equiv N(\varepsilon) = 6 \cdot 5^{t_\varepsilon} - 2$.
- Assuming this conjecture implies that $K(T) \leqslant \frac{4}{3}$ and then also that $K(T) = \frac{4}{3}$.

A Valid Example

With $t_\varepsilon \sim \log(N)$, consider:

$$f(t_\varepsilon) = t_\varepsilon^{\left(\log\left(t_\varepsilon^{\log\left(t_\varepsilon^{\cdot^{\cdot^{\cdot}}}\right)}\right)\right)}$$

where the term $\log(t_\varepsilon)$ is nested n times. Then easily we have

$$\log(f(t_\varepsilon))/t_\varepsilon \sim \frac{(\log(\log N))^{n+1}}{\log N}$$

which decays to 0 for large enough N.

An Invalid Example

On the other hand for a function which grows faster, say

$$f(t_\varepsilon) = t_\varepsilon^{t_\varepsilon}$$

we easily have

$$\log(f(t_\varepsilon))/t_\varepsilon \sim (\log(\log N))$$

which diverges for large enough N.

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