THE SUPPORT OF THE EQUILIBRIUM MEASURE FOR A CLASS OF EXTERNAL FIELDS ON A FINITE INTERVAL

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We investigate the support of the equilibrium measure associated with a class of nonconvex, nonsmooth external fields on a finite interval. Such equilibrium measures play an important role in various branches of analysis. In this paper we obtain a sufficient condition which ensures that the support consists of at most two intervals. This is applied to external fields of the form $-c\,\mathrm{sign}(x)|x|^\alpha$ with $c > 0$, $\alpha \geq 1$ and $x \in [-1, 1]$. If α is an odd integer, these external fields are smooth, and for this case the support was studied before by Deift, Kriecherbauer and McLaughlin, and by Damelin and Kuijlaars.

1. Introduction

In recent years, equilibrium measures with external fields have found an increasing number of applications in a variety of areas. We refer to [**2**, **3**, **4**, **5**, **8**, **10**, **14**, **15**] for these relations, ranging from classical topics as weighted transfinite diameter and weighted Chebyshev polynomials, to more recent developments in weighted approximation, orthogonal polynomials, integrable systems, and random matrix theory.

In the present paper we consider equilibrium problems on the interval $[-1, 1]$. With a continuous function $Q : [-1, 1] \to \mathbb{R}$, we associate the weighted energy of a measure μ on $[-1, 1]$ as follows

$$I_Q(\mu) = \iint \log \frac{1}{|s-t|} d\mu(s) d\mu(t) + 2 \int Q(t) d\mu(t). \tag{1.1}$$

The equilibrium measure in the presence of the external field Q is the unique probability measure μ_Q on $[-1, 1]$ minimizing the weighted energy among all probability measures. Thus

$$I_Q(\mu_Q) = \min\{I_Q(\mu) : \mu \in \mathcal{P}([-1, 1])\} \tag{1.2}$$

where $\mathcal{P}([-1, 1])$ denotes the class

$$\mathcal{P}([-1, 1]) = \{\mu : \mu \text{ is a Borel probability measure on } [-1, 1]\}.$$

The determination of the support of the equilibrium measure is a major step in obtaining the measure. As described by Deift [**2**, Chapter 6] the information that the support consists of N disjoint closed intervals, allows one to set up a system of equations for the endpoints, from which the endpoints may be calculated. Knowing the endpoints, the equilibrium measure may be obtained from a Riemann-Hilbert problem or, equivalently, a singular integral equation.

There are two general useful facts about the equilibrium measure. The first one, due to Mhaskar and Saff [**12**], says that for a convex external field, the support is always one single interval. The other one, due to Deift, Kriecherbauer and McLaughlin [**3**], says that for a real analytic external field, the support always consists of a finite number of intervals. The actual determination of this number is a nontrivial problem. To illustrate the difficulties, Deift, Kriecherbauer and McLaughlin considered explicitly the families of monomial external fields $Q(x) = -cx^n$ with $c \neq 0$, $n \in \mathbb{N}$ and $x \in [-1, 1]$.

In the even case ($n = 2m$) the external field is convex if $c < 0$, and therefore the support is a single interval. For $c > 0$, the external field is concave, and the analysis becomes more involved. Independently from [**3**], this case was considered in [**9**], and it was shown that for every $c > 0$, there are at most three intervals in the support of the equilibrium measure. The same result was also found to be valid for the nonsmooth (i.e. not real analytic) external fields $Q(x) = -c|x|^\alpha$ with $\alpha \geq 1$ not necessarily an even integer.

In the odd case ($n = 2m + 1$) the external field is an odd function, and, by symmetry, we may restrict attention to $c > 0$. In this case the results of [**3**] were extended to the full range of parameters in [**1**]. For all c and all odd integers n, it was shown that the support of the equilibrium measure consists of at most two intervals.

It is the aim of the present paper to study the nonsmooth analogues of $-cx^{2m+1}$ given by

$$Q_{\alpha,c}(x) := -c\,\mathrm{sign}(x)|x|^\alpha = \begin{cases} c|x|^\alpha & \text{for } x \in [-1, 0], \\ -cx^\alpha & \text{for } x \in [0, 1], \end{cases} \tag{1.3}$$

with a real number $\alpha \geq 1$ and $c > 0$. The functions (1.3) are both nonconvex and nonsmooth, and therefore it is of interest to develop methods to determine the nature of the support of the equilibrium measures associated with these external fields.

Our first theorem presents a sufficient condition which ensures that the support of the equilibrium measure is the union of at most two intervals. To formulate it, we use $C^{1+\varepsilon}([-1, 1])$ to denote the class of differentiable functions on $[-1, 1]$, whose derivative satisfies a Hölder condition for some positive exponent. Thus $Q \in C^{1+\varepsilon}([-1,1])$ if and only if

$$|Q'(x) - Q'(y)| \leq C|x-y|^{\varepsilon}, \qquad x,y \in [-1,1]$$

for some $\varepsilon > 0$ and some positive constant C independent of x and y.

Theorem 1.1. *Let $Q \in C^{1+\varepsilon}([-1,1])$. Suppose that there exists a number $a_1 \in [-1,1]$ such that*

(a) *Q is convex on $[-1, a_1]$, and*

(b) *for every $a \in [-1, a_1]$, there is $t_0 \in [a_1, 1]$, such that the function*

$$t \mapsto \frac{1}{\pi} \int_a^1 \frac{Q'(s)}{s-t} \sqrt{(1-s)(s-a)}\, ds \tag{1.4}$$

is non-increasing on (a_1, t_0) and non-decreasing on $(t_0, 1)$. The integral in (1.4) is a principal value integral.

Then $\operatorname{supp}(\mu_Q)$ *is the union of at most two intervals.*

REMARK 1.2. For the special case $a_1 = -1$, Theorem 1.1 was given already in [**9**, Theorem 2].

In our second main result we show that the conditions of Theorem 1.1 are satisfied for the external fields (1.3).

Theorem 1.3. *For $\alpha \geq 1$ and $c > 0$, let $Q_{\alpha,c}$ be given by (1.3). Then for every $a \in [-1, 0]$, there exists $t_0 \in [0,1)$ such that*

$$\frac{1}{\pi} \int_a^1 \frac{Q'_{\alpha,c}(s)}{s-t} \sqrt{(1-s)(s-a)}\, ds \tag{1.5}$$

decreases on $(0, t_0)$ and increases on $(t_0, 1)$. As a result, the support of $\mu_{Q_{\alpha,c}}$ consists of at most two intervals.

REMARK 1.4. For α an odd integer, Theorem 1.3 was established in [**1**]. The proof for this special case differs from the one given here in several respects. For example, the function (1.5) is a polynomial in t whenever α is an odd integer. The proof of the decreasing/increasing property of (1.5) was based in [**1**] on the calculation of the polynomial coefficients and the Descartes' rule of signs for polynomials.

Another difference between [**1**] and the present paper is that in [**1**] the problem was viewed in terms of the parameter c. Quite complicated perturbation arguments were used to obtain from the decreasing/increasing property of (1.5) the conclusion that the support consists of at most two intervals. Here we use Theorem 1.1 and this simplifies the arguments considerably, also in the case where α is an odd integer.

REMARK 1.5. To view the problem in terms of the parameter c is quite natural, since there is a monotonicity with respect to c. To be precise, if Q is fixed then the support $\operatorname{supp}(\mu_{cQ})$ is decreasing as c increases, see

[**1**] or [**14**]. Using this, we can show the following behavior of the support depending on the parameter in case $\alpha > 1$. There exist three critical values $0 < c_1 < c_2 < c_3$ depending on α such that:

(a) For $0 < c \le c_1$, the support $\operatorname{supp}(\mu_{Q_{\alpha,c}})$ is equal to the full interval $[-1,1]$.

(b) For $c_1 < c \le c_2$, there exists $a \in (-1,0)$ such that

$$\operatorname{supp}(\mu_{Q_{\alpha,c}}) = [a,1].$$

(c) For $c_2 < c < c_3$, there exist a_1, b_1 and a_2 such that $-1 < a_1 < b_1 < a_2 < 1$, $a_1 < 0$, and

$$\operatorname{supp}(\mu_{Q_{\alpha,c}}) = [a_1,b_1] \cup [a_2,1].$$

(d) For $c \ge c_3$, there exists $a \in (0,1)$ such that

$$\operatorname{supp}(\mu_{Q_{\alpha,c}}) = [a,1].$$

See [**1**, Theorem 1.1] where this was shown for odd integers $\alpha \ge 3$.

Note that for $\alpha = 1$, the external field (1.3) is convex and the support of $\mu_{Q_{1,c}}$ is an interval containing 1 for every $c > 0$.

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2. The Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1.

2.1. Preliminaries. Let $Q \in C^{1+\varepsilon}([-1,1])$ be fixed. The equilibrium measure μ_Q is characterized by the Euler-Lagrange variational conditions associated with the extremal problem (1.2), which are

$$U^\mu(x) + Q(x) = F \qquad \text{for } x \in \operatorname{supp}(\mu), \tag{2.1}$$

$$U^\mu(x) + Q(x) \ge F \qquad \text{for } x \in [-1,1], \tag{2.2}$$

where F is a constant and

$$U^\mu(x) = \int \log \frac{1}{|x-t|}\, d\mu(t) \tag{2.3}$$

denotes the logaritmic potential of μ, see [**2**, **14**]. The equilibrium measure μ_Q is the only measure from $\mathcal{P}([-1,1])$ satisfying (2.1) and (2.2) for some constant F.

If $\operatorname{supp}(\mu_Q) = \Sigma$ and if μ_Q has a density v, then the equation (2.1) yields

$$\int_\Sigma \log|x-t|\, v(t)dt = Q(x) - F, \qquad x \in \Sigma. \tag{2.4}$$

Then there is a unique constant F, such that the integral equation (2.4) has a solution $v(t)$ satisfying

$$\int_\Sigma v(t)dt = 1. \tag{2.5}$$

If Σ consists of a finite number of nondegenerate closed intervals, then (2.4) may be differentiated for x in the interior of Σ (since Q' is Hölder continuous) to give the singular integral equation

$$\fint_\Sigma \frac{v(t)}{x-t}dt = Q'(x), \qquad x \in \operatorname{int}\Sigma. \tag{2.6}$$

It is well-known, see [**7**, §42.3], that the general solution of (2.6) depends on N parameters, where N is the number of intervals in Σ. These parameters are uniquely determined by the normalization (2.5) and the conditions that the constant F in (2.4) should be the same on each interval of Σ. We also recall that the solutions of (2.6) are Hölder continuous on the interior of Σ, and may become unbounded at endpoints of Σ, cf. [**7**, §5, §42.3].

If we do not know that Σ is the support of μ_Q, we can still consider the function $v(t)$ determined by equations (2.4) and (2.5). Then in general the function $v(t)$ will not be non-negative on Σ. Thus $v(t)$ is the density of a signed measure η that depends on Σ:

$$d\eta(t) = d\eta_\Sigma(t) = v(t)dt.$$

The signed measure η_Σ satisfies

$$\operatorname{supp}(\eta_\Sigma) \subset \Sigma, \qquad \int d\eta_\Sigma = 1, \tag{2.7}$$

and it minimizes the weighted energy $I_Q(\eta)$ among all signed measures satisfying (2.7).

For the special case $\Sigma = [a,1]$, with $a \in [-1,1)$, we have that

$$\frac{d\eta_\Sigma}{dt} = v(t) = \frac{1}{\pi\sqrt{(1-t)(t-a)}}\left[1+G(t)\right], \qquad a < t < 1, \tag{2.8}$$

with

$$G(t) = \frac{1}{\pi}\int_a^1 \frac{Q'(s)}{s-t}\sqrt{(1-s)(s-a)}ds, \tag{2.9}$$

see [**7**, §42.3] or [**16**, §4.3]. Note that (2.9) is equal to the function from (1.4).

Next, we recall the notion of balayage of a measure. The balayage of a nonnegative measure ν with compact support and continuous potential onto a set Σ of positive capacity, is the unique measure $\hat{\nu}$ such that $\operatorname{supp}(\hat{\nu}) \subset \Sigma$, $\|\nu\| = \|\hat{\nu}\|$ and for some constant c,

$$U^{\hat{\nu}}(x) = U^\nu(x) + c, \qquad \text{for quasi every } x \in \Sigma. \tag{2.10}$$

Here quasi every means with the possible exception of a set of capacity zero. We refer the reader to [**11**, **13**, **14**] for these and other notions from logarithmic potential theory. Instead of $\hat{\nu}$ we also write $Bal(\nu;\Sigma)$. For a signed measure ν with Jordan decomposition $\nu = \nu^+ - \nu^-$, the balayage of ν onto Σ is

$$Bal(\nu;\Sigma) = Bal(\nu^+;\Sigma) - Bal(\nu^-;\Sigma)$$

provided the balayages of ν^+ and ν^- exist.

From their defining properties it is then easy to see that the measures η_Σ are related by balayage. That is, if $\Sigma_1 \subset \Sigma_2$, then

$$\eta_{\Sigma_1} = Bal(\eta_{\Sigma_2};\Sigma_1). \tag{2.11}$$

The following result will be used in the proof of Theorem 1.1 below. We say that two sets A and B are quasi-equal, if $A \setminus B$ and $B \setminus A$ have capacity zero.

Lemma 2.1. *Let Σ and Σ_n, $n \in \mathbb{N}$, be closed subsets of $[-1,1]$ having positive capacity such that*

$$\Sigma = \bigcap_n \overline{\bigcup_{k\geq n} \Sigma_k} \tag{2.12}$$

and Σ is quasi-equal to

$$\bigcup_n \bigcap_{k\geq n} \Sigma_k. \tag{2.13}$$

Then the following hold.

(a) *For every finite measure ν with compact support and continuous potential, we have*

$$\lim_{n\to\infty} Bal(\nu;\Sigma_n) = Bal(\nu;\Sigma)$$

with convergence in the sense of weak convergence of measures on $[-1,1]$.*

(b) *If Σ and Σ_n, $n \in \mathbb{N}$, are finite unions of closed intervals, then*

$$\lim_{n\to\infty} \eta_{\Sigma_n} = \eta_\Sigma$$

in the sense of weak convergence of signed measures.*

Proof. (a) Let us write $\nu_n = Bal(\nu;\Sigma_n)$. Then by (2.10), we have for some constant c_n,

$$U^{\nu_n}(x) = U^{\nu}(x) + c_n \qquad \text{for quasi every } x \in \Sigma_n.$$

By weak* compactness, we may assume that (ν_n) converges, say with weak* limit ν^*. Then $\|\nu^*\| = \|\nu\|$ and because of (2.12) we have $\operatorname{supp}(\nu^*) \subset \Sigma$. The lower envelope theorem [**14**] says that

$$U^{\nu^*}(x) = \liminf_{n\to\infty} U^{\nu_n}(x) \qquad \text{for quasi every } x \in \mathbb{C}.$$

Since Σ is quasi-equal to (2.13) it then follows that

$$U^{\nu^*}(x) = U^{\nu}(x) + \liminf_{n\to\infty} c_n \qquad \text{for quasi every } x \in \Sigma.$$

Then $\liminf c_n$ is finite and it follows that ν^* is the balayage of ν onto Σ.

(b) Let $\eta_0 = \eta_{[-1,1]}$. The positive and negative parts of η_0 in the Jordan decomposition $\eta_0 = \eta_0^+ - \eta_0^-$ are compactly supported. They also have continuous potentials. Indeed, the function G from (2.9) (with $a = -1$) is continuous, and therefore it is bounded on $[-1,1]$. Then it follows from the representation (2.8)–(2.9) for η_0, that both η_0^+ and η_0^- are bounded above by a constant times the measure $1/(\pi\sqrt{1-t^2})dt$. This measure has a continuous potential — in fact its potential is constant on $[-1,1]$ — and therefore the potentials of η_0^+ and η_0^- are continuous as well, see [**6**, Lemma 5.2]. Thus it follows from part (a) that

$$Bal(\eta_0^+;\Sigma_n) \xrightarrow{*} Bal(\eta_0^+;\Sigma)$$

and

$$Bal(\eta_0^-;\Sigma_n) \xrightarrow{*} Bal(\eta_0^-;\Sigma).$$

Then

$$Bal(\eta_0;\Sigma_n) \xrightarrow{*} Bal(\eta_0;\Sigma).$$

Since η_Σ is equal to the balayage of η_0 onto Σ, and similarly η_{Σ_n} is the balayage of η_0 onto Σ_n, part (b) follows. □

2.2. A lemma on convexity. The convexity assumption (a) of Theorem 1.1 will be used via the following lemma.

Lemma 2.2. *Let $Q \in C^{1+\varepsilon}([-1,1])$. Let $\Sigma \subset [-1,1]$ be a finite union of nondegenerate closed intervals. Let $\eta = \eta_\Sigma$ be the signed measure associated with Σ, as described in Section* 2.1*, and let v be the density of η. Suppose that $[a,b] \subset \Sigma$ and that*

(a) *Q is convex on $[a,b]$,*
(b) *$v(a) \geq 0$, and $v(b) \geq 0$,*
(c) *$v(t) \geq 0$ on $\Sigma \setminus [a,b]$.*

Then $v(t) > 0$ for all $t \in (a,b)$.

Remark 2.3. The density v is continuous on the interior of Σ, and may become unbounded ($\pm\infty$) at endpoints of Σ. The assumption (b) is also satisfied if $v(a) = +\infty$ in case a is an endpoint, and similarly for b.

Proof. First, we reduce the problem to the case $\Sigma = [a,b]$. Write $\eta = \eta_1 + \eta_2$, where η_1 is the restriction of η to $[a,b]$ and η_2 the restriction to $\Sigma \setminus [a,b]$. From (2.4) we get

$$U^{\eta_1}(x) + Q_1(x) = F \qquad \text{for } x \in [a,b],$$

where

$$Q_1(x) = U^{\eta_2}(x) + Q(x), \qquad x \in [a,b].$$

The measure η_2 is nonnegative by assumption (c). Then it is easy to see from (2.3) that the logarithmic potential U^{η_2} is convex on $[a,b]$. Thus Q_1 is convex on $[a,b]$ because of assumption (a). The potential U^{η_2} is real analytic on the open interval (a,b), and therefore Q_1' satisfies a Hölder condition on (a,b). At the endpoins a and b, Q_1' could have a singularity of logarithmic type, but this will not affect the arguments that follow. In particular, the representation (2.14) below, remains valid, cf. [**16**, §4.3].

Therefore, we have reduced the proof of the lemma to the case when $\Sigma = [a,b]$. Without loss of generality, we may also assume that $[a,b] = [-1,1]$. Then, as in (2.8), the density v is given by

$$v(t) = \frac{1}{\pi\sqrt{1-t^2}}\left[1 + G(t)\right] \tag{2.14}$$

and

$$G(t) = \frac{1}{\pi} \int_{-1}^{1} \frac{Q'(s)}{s-t}\sqrt{1-s^2}ds.$$

In the principal value integral we remove the singular part as follows

$$G(t) = \frac{1}{\pi}\int_{-1}^{1} \frac{Q'(s)-Q'(t)}{s-t}\sqrt{1-s^2}ds + \frac{Q'(t)}{\pi} \int_{-1}^{1} \frac{1}{s-t}\sqrt{1-s^2}ds.$$

The remaining principal value integral we write as

$$\int_{-1}^{1} \frac{1-s^2}{s-t}\frac{ds}{\sqrt{1-s^2}} = \int_{-1}^{1} \frac{(1-s^2)-(1-t^2)}{s-t}\frac{ds}{\sqrt{1-s^2}} = -\int_{-1}^{1}\frac{s+t}{\sqrt{1-s^2}}ds,$$

where we used the fact that

$$\int_{-1}^{1} \frac{1}{s-t}\frac{ds}{\sqrt{1-s^2}} = 0.$$

Thus

$$G(t) = \frac{1}{\pi}\int_{-1}^{1} \frac{Q'(s)-Q'(t)}{s-t}\sqrt{1-s^2}ds - \frac{Q'(t)}{\pi}\int_{-1}^{1}\frac{s+t}{\sqrt{1-s^2}}ds. \tag{2.15}$$

Next, we have that

$$\begin{aligned}\left(\frac{1+t}{2}\right)G(1) + \left(\frac{1-t}{2}\right)G(-1) &= -\left(\frac{1+t}{2}\right)\frac{1}{\pi}\int_{-1}^{1} Q'(s)(1+s)\frac{ds}{\sqrt{1-s^2}} \\ &\quad + \left(\frac{1-t}{2}\right)\frac{1}{\pi}\int_{-1}^{1} Q'(s)(1-s)\frac{ds}{\sqrt{1-s^2}}.\end{aligned}$$

Combining the two integrals, and using

$$-\left(\frac{1+t}{2}\right)(1+s) + \left(\frac{1-t}{2}\right)(1-s) = -(s+t),$$

we obtain

$$\left(\frac{1+t}{2}\right)G(1)+\left(\frac{1-t}{2}\right)G(-1)=-\frac{1}{\pi}\int_{-1}^{1}Q'(s)\frac{s+t}{\sqrt{1-s^2}}ds. \tag{2.16}$$

From (2.15) and (2.16) we learn that

$$\begin{aligned}
&G(t)-\left(\frac{1+t}{2}\right)G(1)-\left(\frac{1-t}{2}\right)G(-1)\\
&=\frac{1}{\pi}\int_{-1}^{1}\frac{Q'(s)-Q'(t)}{s-t}\sqrt{1-s^2}ds-\frac{1}{\pi}\int_{-1}^{1}(Q'(t)-Q'(s))\frac{s+t}{\sqrt{1-s^2}}ds\\
&=\frac{1}{\pi}\int_{-1}^{1}\frac{Q'(t)-Q'(s)}{t-s}\sqrt{1-s^2}ds-\frac{1}{\pi}\int_{-1}^{1}\frac{Q'(t)-Q'(s)}{t-s}\frac{t^2-s^2}{\sqrt{1-s^2}}ds\\
&=\frac{1}{\pi}\int_{-1}^{1}\frac{Q'(t)-Q'(s)}{t-s}\left[\sqrt{1-s^2}-\frac{t^2-s^2}{\sqrt{1-s^2}}\right]ds\\
(2.17)\ &=\frac{1}{\pi}\int_{-1}^{1}\frac{Q'(t)-Q'(s)}{t-s}\frac{1-t^2}{\sqrt{1-s^2}}ds.
\end{aligned}$$

The convexity of Q implies that

$$\frac{Q'(t)-Q'(s)}{t-s}\geq 0$$

for every s and t in $(-1,1)$. Then for $t\in(-1,1)$, the integral (2.17) is non-negative and this proves the inequality

$$G(t)\geq\left(\frac{1+t}{2}\right)G(1)+\left(\frac{1-t}{2}\right)G(-1),\qquad -1<t<1. \tag{2.18}$$

Actually, we have strict inequality in (2.18), unless Q' is a constant. Indeed, if equality holds in (2.18) at a certain $t\in(-1,1)$, then it follows from (2.17) that

$$\frac{Q'(t)-Q'(s)}{t-s}=0$$

for almost all $s\in(-1,1)$. Since Q' is continuous, this can only happen if $Q'(s)=Q'(t)$ for all s, and this means that Q' is constant.

Thus, if Q' is not a constant, we see that

$$G(t)>\left(\frac{1+t}{2}\right)G(1)+\left(\frac{1-t}{2}\right)G(-1),\qquad -1<t<1, \tag{2.19}$$

and then it follows from assumption (b) and (2.14) that $1+G(1)\geq 0$ and $1+G(-1)\geq 0$. The right-hand side of (2.19) is a convex combination of $G(1)$ and $G(-1)$. Thus it follows from (2.19) that $1+G(t)>0$ for all $t\in(-1,1)$. In view of (2.14), we then have $v(t)>0$ in case Q' is not a constant.

If Q' is a constant, say $Q'(t) = k$, then we obtain from (2.15) that $G(t) = -kt$. Hence

$$v(t) = \frac{1-kt}{\pi\sqrt{1-t^2}}, \qquad -1 < t < 1.$$

Then from $v(-1) \geq 0$ and $v(1) \geq 0$, we get $|k| \leq 1$, and then clearly $v(t) > 0$ on $(-1,1)$. This completes the proof of Lemma 2.2. □

2.3. Proof of Theorem 1.1.

Proof. We write $\mu = \mu_Q$. Let us first assume that $\operatorname{supp}(\mu) \subset [a_1, 1]$. From the assumption (b) of Theorem 1.1 with $a = a_1$, we have that there exists $t_0 \in (a_1, 1)$, such that

$$\frac{1}{\pi}\int_{a_1}^{1} \frac{Q'(s)}{s-t}\sqrt{(1-s)(s-a_1)}ds$$

is non-increasing on (a_1, t_0) and non-decreasing on $(t_0, 1)$. As no points of $\operatorname{supp}(\mu)$ lie to the left of a_1, we may apply [**9**, Theorem 2] on the restricted interval $[a_1, 1]$ and deduce that $\operatorname{supp}(\mu)$ is either an interval containing a_1, or an interval containing 1, or the union of an interval containing a_1 with an interval containing 1. This proves the theorem in case the support of μ is contained in $[a_1, 1]$.

For the rest of the proof, we shall assume that $\operatorname{supp}(\mu)$ is not contained in $[a_1, 1]$. Let

$$a := \min\{x : x \in \operatorname{supp}(\mu)\} \tag{2.20}$$

so that $a < a_1$.

For every pair (p, q) with $a < p \leq q \leq 1$, we let $v_{p,q}$ be the density of the signed measure η_Σ with $\Sigma = [a, p] \cup [q, 1]$ if $q < 1$ and $\Sigma = [a, p]$ if $q = 1$. See Section 2.1 for the definition of η_Σ.

We introduce the set Z consisting of all pairs (p, q) satisfying the following four conditions:

(a) $a < p \leq q \leq 1$ and $q \geq a_1$.
(b) $\operatorname{supp}(\mu) \subset [a, p] \cup [q, 1]$.
(c) If $q < 1$ then $\pi\sqrt{(1-t)(t-a)}v_{p,q}(t)$ is non-decreasing for $t \in (q, 1)$.
(d) If $p > a_1$ then $\pi\sqrt{(1-t)(t-a)}v_{p,q}(t)$ is non-increasing for $t \in (a_1, p)$.

We observe first that $Z \neq \emptyset$. Indeed, from the assumption (b) of Theorem 1.1 it follows that there exists $t_0 \in [a_1, 1]$ such that

$$\frac{1}{\pi}\int_{a}^{1} \frac{Q'(s)}{s-t}\sqrt{(1-s)(s-a)}ds, \qquad a < t < 1$$

is non-increasing on (a_1, t_0) and non-decreasing on $(t_0, 1)$. Since for $a < t < 1$, we have

$$\pi\sqrt{(1-t)(t-a)}v_{t_0,t_0}(t) = 1 + \frac{1}{\pi}\int_{a}^{1} \frac{Q'(s)}{s-t}\sqrt{(1-s)(s-a)}ds,$$

by (2.8) and (2.9), we see that properties (c) and (d) are satisfied for the pair (t_0, t_0). Properties (a) and (b) are trivially satisfied, so that (t_0, t_0) belongs to Z. Hence Z is non-empty indeed.

Next, we want to show that Z is closed. To this end, we take $(p, q) \in \bar{Z}$ and we choose sequences (p_n) and (q_n) such that

$$(p_n, q_n) \in Z, \qquad p_n \to p, \qquad q_n \to q.$$

We verify that the properties (a)–(d) hold for the pair (p, q). Since (p_n, q_n) belongs to Z, we have by (b) that $[a, p_n] \cup [q_n, 1]$ contains the support of μ for every n. It then follows that $[a, p] \cup [q, 1]$ contains $\operatorname{supp}(\mu)$. Thus (b) holds. Since $a \in \operatorname{supp}(\mu)$ and $\operatorname{supp}(\mu)$ does not have isolated points, we find that $p > a$. The other inequalities of (a) are immediate. To establish (c) and (d), we first note that by Lemma 2.1 we have in the sense of weak* convergence of signed measures

$$v_{p_n,q_n}(t)dt \overset{*}{\to} v_{p,q}(t)dt. \tag{2.21}$$

Now suppose that (c) does not hold. Then there exist t_1 and t_2 with $q < t_1 < t_2 < 1$ such that

$$\pi\sqrt{(1-t_1)(t_1-a)}v_{p,q}(t_1) > \pi\sqrt{(1-t_2)(t_2-a)}v_{p,q}(t_2).$$

Since v is continuous, there exists $\varepsilon > 0$ such that

$$\pi \int_{t_1-\varepsilon}^{t_1+\varepsilon} \sqrt{(1-t)(t-a)}v_{p,q}(t)dt > \pi \int_{t_2-\varepsilon}^{t_2+\varepsilon} \sqrt{(1-t)(t-a)}v_{p,q}(t)dt.$$

We may assume that ε is chosen sufficiently small so that $[t_1 - \varepsilon, t_1 + \varepsilon]$ and $[t_2 - \varepsilon, t_2 + \varepsilon]$ are disjoint intervals that are both contained in $(q, 1)$. From the weak* convergence (2.21) it then easily follows that we must have for n large enough,

$$\pi \int_{t_1-\varepsilon}^{t_1+\varepsilon} \sqrt{(1-t)(t-a)}v_{p_n,q_n}(t)dt > \pi \int_{t_2-\varepsilon}^{t_2+\varepsilon} \sqrt{(1-t)(t-a)}v_{p_n,q_n}(t)dt.$$

For n large enough, we also have $q_n < t_1 - \varepsilon$. Then we arrive at a contradiction, since (c) holds for the pair (p_n, q_n). Thus property (c) holds for the pair (p, q). In a similar way, it follows that (d) holds. Therefore Z is a closed set.

Since Z is a closed non-empty subset of $[a, 1] \times [a_1, 1]$, we can find a pair in Z for which the difference $q - p$ is maximal. Such a maximizer may not be unique (when we have finished the proof, we will see that it is), but we take any such pair and denote it by (p^*, q^*). Let $\Sigma = [a, p^*] \cup [q^*, 1]$ in case $q^* < 1$, and $\Sigma = [a, p^*]$ in case $q^* = 1$. For short, we write v^* instead of v_{p^*,q^*}. Our aim is to show that $\operatorname{supp}(\mu) = \Sigma$. Having established that, it will follow from the uniqueness of μ that (p^*, q^*) is the only maximizer for the difference $q - p$. We prove that $\operatorname{supp}(\mu) = \Sigma$ by showing that v^* is positive on the interior of Σ.

We consider several cases. First we assume $q^* < 1$ and we consider the interval $(q^*, 1)$. Suppose that v^* is nonpositive somewhere on $(q^*, 1)$. Then by property (c) it follows that there exists $\varepsilon \in (0, 1-q^*)$ such that v^* is nonpositive on $[q^*, q^*+\varepsilon]$. We claim that $(p^*, q^*+\varepsilon)$ satisfies the conditions (a)–(d). It is clear that (a) is satisfied. For (b), we recall from [**9**, Lemma 3] that

$$\operatorname{supp}(\mu) \subset \overline{\{x : v^*(x) > 0\}} \subset [a, p^*] \cup [q^* + \varepsilon, 1].$$

To see (c) and (d), we note that $v_{p^*, q^*+\varepsilon}$ is obtained from v^* by taking the balayage of v^* onto $[a, p^*] \cup [q^* + \varepsilon, 1]$. Since v^* is nonpositive on the gap $(p^*, q^* + \varepsilon)$, we see using [**9**, Lemma 4 (2)], that this process preserves the properties (c) and (d). Thus $(p^*, q^* + \varepsilon) \in Z$. However, this contradicts the maximality of $q^* - p^*$. Thus our assumption that v^* is nonpositive somewhere in $(q^*, 1)$ is incorrect, and it follows that v^* is positive on the interval $(q^*, 1)$.

Now consider the case $p^* > a_1$. In a similar way as above it follows that v^* is positive on (a_1, p^*). Because of property (d) and the continuity of v^*, we find $v^*(a_1) > 0$. Since

$$\operatorname{supp}(\mu) \subset \overline{\{x : v^*(x) > 0\}},$$

see [**9**, Lemma 3], and $a \in \operatorname{supp}(\mu)$, we also have $v^*(a) \geq 0$. Since Q is convex on $[a, a_1]$ and $v^* \geq 0$ outside $[a, a_1]$, it follows from Lemma 2.2 that $v^* > 0$ on (a, a_1). So we have shown that $v^* > 0$ on the interval (a, p^*) in case $p^* > a_1$.

What remains is the case $p^* \leq a_1$. If $v^*(p^*) < 0$, then v^* is negative on $[p^* - \varepsilon, p^*]$ for some $\varepsilon > 0$ with $\varepsilon < p^* - a$. Then we may take the balayage of this negative part onto $[a, p^* - \varepsilon] \cup [q^*, 1]$ and it follows as above that the pair $(p^* - \varepsilon, q^*)$ belongs to Z. This is a contradiction. Thus $v^*(p^*) \geq 0$. Since Q is convex on $[a, p^*]$ with $v^*(a) \geq 0$, $v^*(p^*)$, and $v^* \geq 0$ outside of $[a, p^*]$, it follows again from Lemma 2.2 that v^* is positive on (a, p^*).

Thus in both cases, we find that $v^* > 0$ on (a, p^*). We also proved that $v^* > 0$ on $(q^*, 1)$ in case $q^* < 1$. Thus v^* is positive on the interior of Σ. It follows that $\operatorname{supp}(\mu) = \Sigma$. This completes the proof of Theorem 1.1, since Σ is the union of at most two intervals. □

3. The Proof of Theorem 1.3

Proof. We write $Q = Q_{\alpha,c}$. Clearly Q is convex on $[-1, 0]$. Let us set for $a \in [-1, 0]$ and for $t \in [0, 1]$,

$$G_\alpha(t) := \frac{1}{c\alpha\pi}\int_a^1 \frac{Q'(s)}{s-t}\sqrt{(1-s)(s-a)}ds \tag{3.1}$$
$$= -\frac{1}{\pi}\int_a^0 \frac{|s|^{\alpha-1}\sqrt{(1-s)(s-a)}}{s-t}ds$$
$$-\frac{1}{\pi}\int_0^1 \frac{s^{\alpha-1}\sqrt{(1-s)(s-a)}}{s-t}ds$$
$$=: I_1 + I_2.$$

Here the second integral I_2 is a principal value integral.

We have to prove that there exists $t_\alpha \in [0,1)$ so that G_α decreases in $(0,t_\alpha)$ and increases in $(t_\alpha,1)$ (if $t_\alpha = 0$ then the first condition is an empty one). We establish the following properties:

(i) $G_\alpha(0) \le 0$;
(ii) $G_\alpha(1) > 0$;
(iii) For every $\alpha > 1$, there is $t_\alpha \in [0,1)$, such that $G'_\alpha(t) < 0$ on $(0,t_\alpha)$, $G'_\alpha(t) > 0$ on $(t_\alpha,1)$, and $G''_\alpha(t) \ge 0$ on $(t_\alpha,1)$.

Clearly, then (iii) implies the decreasing/increasing property of G_α.

To show (i), we write

$$G_\alpha(0) = -\frac{1}{\pi}\left[\int_0^1 s^{\alpha-2}\sqrt{(1-s)(s-a)}ds - \int_a^0 |s|^{\alpha-2}\sqrt{(1-s)(s-a)}ds\right]$$

and in the second integral we make the change of variables $s \mapsto as$, to find

$$G_\alpha(0) = -\frac{1}{\pi}\int_0^1 s^{\alpha-2}\sqrt{1-s}\left(\sqrt{s-a} - |a|^{\alpha-\frac{1}{2}}\sqrt{1-as}\right)ds.$$

Since $\sqrt{s-a}$ is greater than or equal to $|a|^{\alpha-\frac{1}{2}}\sqrt{1-as}$ for s in the interval $[0,1]$, we find that $G_\alpha(0) \le 0$, as claimed in (i). Note that $G_\alpha(0) = 0$ if and only if $a = -1$.

Next, it is easy to see from (3.1) that

$$G_\alpha(1) = \frac{1}{\pi}\int_a^1 \frac{|s|^{\alpha-1}\sqrt{s-a}}{\sqrt{1-s}}ds > 0, \tag{3.2}$$

which establishes (ii) for all $\alpha \ge 1$.

We now prove (iii) by induction on $k = [\alpha]$, where $[\alpha]$ denotes the integer part of α.

For $\alpha = 1$, we find by explicit calculation

$$G_1(t) = -\frac{1}{\pi}\int_a^1 \frac{\sqrt{(1-s)(s-a)}s-t}{d}s = t - \frac{1+a}{2}.$$

Then (iii) is satisfied with $t_1 = t_\alpha = 0$. Suppose now $1 < \alpha < 2$. Consider the analytic function

$$f(z) := z^{\alpha-1}\left[(z-1)(z-a)\right]^{1/2}$$

defined for $z \in \mathbb{C} \setminus (-\infty, 1]$, where that branch of the square root is chosen which is positive for real $z > 1$. Then the principal value integral I_2 may be written as

$$I_2 = -\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-t}dz$$

with the contour γ going from 0 to 1 on the upper side of the cut $(-\infty, 1]$ and back from 1 to 0 on the lower side.

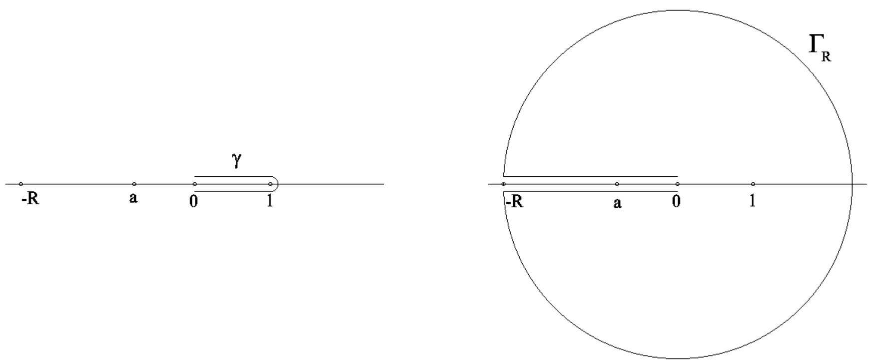

Figure 1. The contours γ and Γ_R.

We transform γ into the contour Γ_R going from 0 to $-R$ on the upper side of the cut, continuing along the big circle C_R of radius R going to $-R$ on the lower side of the cut, and then going from $-R$ to 0 on the lower side of the cut. We choose $R > 1$. See Figure 1 for γ and Γ_R.

The contribution from the upper and lower sides of the cut comes from the imaginary part of f, which is

$$\Im f(x+i0) = \begin{cases} \sin(\alpha\pi)|x|^{\alpha-1}\sqrt{(1-x)(a-x)} & \text{for } x < a, \\ -\cos(\alpha\pi)|x|^{\alpha-1}\sqrt{(1-x)(x-a)} & \text{for } a < x < 0, \end{cases}$$

and $\Im f(x-i0) = -\Im f(x+i0)$. Therefore

$$\begin{aligned} I_2 &= -\frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z-t}dz \\ &\quad +\frac{\sin\alpha\pi}{\pi}\int_{-R}^{a}\frac{|x|^{\alpha-1}\sqrt{(1-x)(a-x)}}{x-t}dx \\ &\quad -\frac{\cos\alpha\pi}{\pi}\int_{a}^{0}\frac{|x|^{\alpha-1}\sqrt{(1-x)(x-a)}}{x-t}dx. \end{aligned}$$

Thus we have shown that

$$\begin{aligned} G_\alpha(t) &= -\frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z-t}dz \\ &\quad +\frac{\sin\alpha\pi}{\pi}\int_{-R}^{a}\frac{|x|^{\alpha-1}\sqrt{(1-x)(a-x)}}{x-t}dx \\ &\quad -\left(\frac{1+\cos\alpha\pi}{\pi}\right)\int_{a}^{0}\frac{|x|^{\alpha-1}\sqrt{(1-x)(x-a)}}{x-t}dx. \end{aligned} \tag{3.3}$$

From (3.3), we obtain for the second derivative

$$\begin{aligned} G''_\alpha(t) &= -\frac{1}{\pi i}\int_{C_R}\frac{f(z)}{(z-t)^3}dz \\ &\quad +\frac{2\sin\alpha\pi}{\pi}\int_{-R}^{a}\frac{|x|^{\alpha-1}\sqrt{(1-x)(a-x)}}{(x-t)^3}dx \\ &\quad -\frac{2\,(1+\cos\alpha\pi)}{\pi}\int_{a}^{0}\frac{|x|^{\alpha-1}\sqrt{(1-x)(x-a)}}{(x-t)^3}dx. \end{aligned} \tag{3.4}$$

We let $R\to\infty$ in (3.4). Then the integral over the circle C_R tends to 0, since the integrand behaves like $z^{\alpha-3}$ as $|z|\to\infty$. Then we get the representation

$$\begin{aligned} G''_\alpha(t) &= \frac{2\sin\alpha\pi}{\pi}\int_{-\infty}^{a}\frac{|x|^{\alpha-1}\sqrt{(1-x)(a-x)}}{(x-t)^3}dx \\ &\quad -\frac{2\,(1+\cos\alpha\pi)}{\pi}\int_{a}^{0}\frac{|x|^{\alpha-1}\sqrt{(1-x)(x-a)}}{(x-t)^3}dx. \end{aligned} \tag{3.5}$$

Note that the improper integral is convergent because $\alpha<2$. Since $1<\alpha<2$, we have $\sin\alpha\pi<0$. Also $(x-t)^3<0$ whenever $x<0<t$. Thus we conclude that

$$G''_\alpha(t) > 0, \qquad \text{for } t\in(0,1), \tag{3.6}$$

in case $1<\alpha<2$. Thus G_α is strictly convex on $(0,1)$. Since $G_\alpha(0)<G_\alpha(1)$ by properties (i) and (ii) the property (iii) follows for $\alpha\in(1,2)$. Thus we have established (iii) whenever $k=[\alpha]=1$.

Now let $k \geq 2$ and suppose that (iii) is true for all α with $[\alpha] = k - 1$. Let $\alpha \in [k, k+1)$. From (3.1) we obtain for $0 < t < 1$,

$$\begin{aligned} G_\alpha(t) &= -\frac{1}{\pi}\int_a^1 \frac{|x|^{\alpha-1} - t|x|^{\alpha-2} + t|x|^{\alpha-2}}{x-t}\sqrt{(1-x)(x-a)}\,dx \\ &= -\frac{1}{\pi}\int_a^1 \frac{|x|^{\alpha-1} - t|x|^{\alpha-2}}{x-t}\sqrt{(1-x)(x-a)}\,dx + tG_{\alpha-1}(t) \\ &=: F(t) + tG_{\alpha-1}(t). \end{aligned} \tag{3.7}$$

We can write that

$$\begin{aligned} F(t) &:= -\frac{1}{\pi}\int_a^1 \frac{|x|-t}{x-t}|x|^{\alpha-2}\sqrt{(1-x)(x-a)}\,dx \\ &= \frac{1}{\pi}\int_a^0 \frac{x+t}{x-t}|x|^{\alpha-2}\sqrt{(1-x)(x-a)}\,dx \\ &\quad -\frac{1}{\pi}\int_0^1 |x|^{\alpha-2}\sqrt{(1-x)(x-a)}\,dx, \end{aligned}$$

from which we obtain

$$F'(t) = \frac{1}{\pi}\int_a^0 \frac{2x}{(x-t)^2}|x|^{\alpha-2}\sqrt{(1-x)(x-a)}\,dx < 0, \qquad 0 < t < 1 \tag{3.8}$$

and

$$F''(t) = \frac{1}{\pi}\int_a^0 \frac{4x}{(x-t)^3}|x|^{\alpha-2}\sqrt{(1-x)(x-a)}\,dx > 0, \qquad 0 < t < 1. \tag{3.9}$$

Differentiating (3.7) we get

$$G'_\alpha(t) = F'(t) + G_{\alpha-1}(t) + tG'_{\alpha-1}(t) \tag{3.10}$$

and

$$G''_\alpha(t) = F''(t) + 2G'_{\alpha-1}(t) + tG''_{\alpha-1}(t). \tag{3.11}$$

By the inductive hypothesis, there exists $t_{\alpha-1}$, such that $G'_{\alpha-1}(t)$ is negative on $(0, t_{\alpha-1})$ and positive on $(t_{\alpha-1}, 1)$, as well as $G''_{\alpha-1}(t) \geq 0$ on $(t_{\alpha-1}, 1)$.

Suppose first that $t_{\alpha-1} > 0$. Since $G_\alpha(0) \leq 0$ and $G'_{\alpha-1}(t) < 0$ on $(0, t_{\alpha-1})$, we have that $G_{\alpha-1}(t)$ is strictly decreasing on $(0, t_{\alpha-1})$, and therefore is negative there. This, together with (3.8) and (3.10), implies that $G'_\alpha(t) < 0$ on $(0, t_{\alpha-1}]$. On the other hand from (3.9), (3.11) and the inductive hypothesis, we obtain that $G''_\alpha(t) > 0$ on $[t_{\alpha-1}, 1)$. This implies that G_α is strictly convex on $(t_{\alpha-1}, 1)$. Since G_α and G'_α are negative on $(0, t_{\alpha-1}]$, and $G_\alpha(1) > 0$, we see that there exists $t_\alpha \in (t_{\alpha-1}, 1)$, such that $G'_\alpha(t)$ is negative on $(0, t_\alpha)$ and positive on $(t_\alpha, 1)$. It is clear also that $G''_\alpha(t) > 0$ on $(t_\alpha, 1)$. Thus property (iii) holds in case $t_{\alpha-1} > 0$.

If $t_{\alpha-1} = 0$, then we still use (3.9) and (3.11) to derive $G''_\alpha(t) > 0$ on $(0, 1)$, which implies that G_α is strictly convex on $[0, 1]$. Since $G_\alpha(0) < G_\alpha(1)$, the property (iii) follows as well.

The property (iii) is now established whenever $[\alpha] = k$. By induction we derive that it is true for every $k \geq 1$, that is it holds for every $\alpha \geq 1$. Thus there exists $t_0 \in [0, 1)$ such that (1.5) decreases on $(0, t_0)$ and increases on $(t_0, 1)$. Since Q is convex on $[-1, 0]$, the conditions of Theorem 1.1 are satisfied with $a_1 = 0$. It follows from Theorem 1.1 that the support of the equilibrium measure consists of at most two intervals. □

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