ON SUPPORTS OF EQUILIBRIUM MEASURES WITH CONCAVE SIGNED EQUILIBRIA

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Abstract. Let $\Sigma \subset \mathbb{R}$ be compact. In this paper, we study the support of the equilibrium measure for a class of external fields $Q : \Sigma \to \mathbb{R}$, whose associated signed equilibrium measure has a positive part with concave density supported on at most two intervals. We prove that the support of the equilibrium measure is at most two intervals. Our proof uses the iterated balayage algorithm. As a corollary we obtain by a constructive method that the equilibrium measure of any two intervals has convex density. A non-trivial counterpart of the results to the unit circle is also presented.

Key words. Logarithmic potential theory, external fields, equilibrium measure, equilibrium support

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1. Introduction.

In recent years minimal energy problems with external fields have found many applications in a variety of areas ranging from diverse subjects such as orthogonal polynomials, weighted Fekete points, numerical conformal mappings, weighted polynomial approximation, rational and Páde approximation, integrable systems, random matrix theory and random permutations. Let $\Sigma \subset \mathbb{C}$ be closed. An essential step towards the solution of such minimal energy problems is the determination of the nature of the support of the equilibrium measure $\mu := \mu_Q$ associated with a given external field $Q : \Sigma \to (-\infty, \infty]$. As described by Deift [8, Chapter 6], information that the support consists of finitely many intervals allows one to set up a system of equations for the endpoints, from which the endpoints may be calculated, and thus, the equilibrium measure may be obtained from a Riemann-Hilbert problem or, equivalently, a singular integral equation. It is for this reason, that it is important to have a priori, conditions to ensure that the support S of μ is the union of a finite number of intervals. For more on the various applications of minimal energy problems we refer the reader to the references [1, 2, 3, 6, 7, 8, 9, 15, 17, 18, 19, 20] and those listed there in.

In this paper we establish a sufficient condition that the equilibrium support consists of at most two intervals. The method utilizes the Iterated Balayage Algorithm, introduced and used first in the papers [15, 6, 7]. As a result from our analysis we establish also that the equilibrium measure for any two intervals is convex. In order to formulate our results we introduce some needed definitions and notations from potential theory.

1.1. Potential-Theoretical Preliminaries.

Let $\Sigma = \cup_{i=1}^{n}[a_i, b_i]$ or let Σ be finitely many closed arcs on the unit circle. Let $w = \exp(-Q)$ be a function (called weight), where $Q : \Sigma \to (-\infty, \infty]$ is continuous in an extended sense but not identically infinity.

Given a Borel probability measure ν supported on Σ, its logarithmic potential and logarithmic energy are respectively given by

$$U^{\nu}(z) := \int \log \frac{1}{|z-t|} d\nu(t), \quad I(\nu) := \int \int \log \frac{1}{|s-t|} d\nu(s) d\nu(t).$$

The weighted energy of ν associated with the external field Q (or weight w) is defined as

$$I_w(\nu) := I(\nu) + 2 \int Q(x)\, d\nu(x).$$

It is well known (see [11], [19, Theorem 1.3]), that under the above assumptions there exists a unique *equilibrium measure* $\mu := \mu_w$ associated with Q, that solves the minimal energy problem

$$I_w(\mu) = \mathcal{E}_w := \min_{\nu \in \mathcal{P}(\Sigma)} I_w(\nu) \tag{1.1}$$

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where $\mathcal{P}(\Sigma)$ denotes the class of all Borel probability measures supported on Σ. The support of the equilibrium measure is denoted by S_μ or S_Q. This measure is completely characterized by the Gauss variational conditions

$$\begin{cases} U^\mu(x) + Q(x) = F, & x \in S_Q, \\ U^\mu(x) + Q(x) \geq F, & x \in \Sigma, \end{cases} \tag{1.2}$$

with some constant F. We note that under the assumptions we have, the logarithmic potential $U^\mu(z)$ is continuous in $\mathbb{C}$. In the particular case when $Q \equiv 0$ the unique minimizer μ_Σ of (1.1) is called an *equilibrium measure of* Σ.

Next we introduce the notion of *signed equilibrium* (see [4]).

DEFINITION 1. *Given a compact subset $E \subset \mathbb{C}$ and an external field Q, we call a signed measure η_E supported on E, and of total mass $\eta_E(E) = 1$, a signed equilibrium on E associated with Q, if*

$$U^{\eta_E}(x) + Q(x) = F_E \qquad \text{for all } x \in E. \tag{1.3}$$

The choice of the normalization $\eta_E(E) = 1$ is just for convenience in the applications here.

The next subsection is a brief summary of an essential tool in our analysis, called the Iterated Balayage Algorithm (see [15, 6, 7]). For the rest of the paper we will assume that Σ is a union of finitely many intervals and that the external field Q has δ-Hölder continuous first derivative for some $\delta > 0$.

1.2. The Iterated Balayage Algorithm (IBA). We recall the notion of *balayage* onto a compact set (see [16, Chapter IV]). Let M be a compact subset of the complex plane with positive logarithmic capacity and such that the complement $\overline{\mathbb{C}} \setminus M$ is regular. Then, if ν is any finite positive Borel measure on $\mathbb{C}$ with compact support, there exists a unique measure $\hat{\nu}$ supported on M such that $\|\nu\| = \|\hat{\nu}\|$, and for some constant C,

$$U^{\hat{\nu}}(z) = U^\nu(z) + C, \qquad z \in M.$$

The measure $\hat{\nu}$ is called the balayage of ν onto M and we denote it by $\mathrm{Bal}(\nu; M)$. For a signed measure $\sigma = \sigma^+ - \sigma^-$, the balayage is defined as $\mathrm{Bal}(\sigma; M) := \mathrm{Bal}(\sigma^+; M) - \mathrm{Bal}(\sigma^-; M)$.

The *Iterated Balayage Algorithm (IBA)*, presents an iterative method to solve the variational problem (1.2). Given an external field Q on $\Sigma_0 := \Sigma$ one proceeds as follows. The first step is to solve the integral equation

$$\int_{\Sigma_0} \log|x-t| v_0(t)dt = Q(x) - F_0, \qquad x \in \Sigma_0, \tag{1.4}$$

subject to the condition

$$\int_{\Sigma_0} v_0(t)dt = 1. \tag{1.5}$$

Here, F_0 is a fixed constant.

Since $Q \in C^{1+\delta}(\Sigma_0)$ (recall that Σ_0 consists of finitely many intervals), it can be shown that (1.4)–(1.5) has a unique solution $v_0(t)$, given by the Sokhotski-Plemelj formula (see [12, p. 425]). In the simplest case when $\Sigma_0 = [a, b]$ the formula for v_0 is (see [12, p. 428]),

$$v(x) = v_0(x) := \frac{1}{\pi\sqrt{(b-x)(x-a)}} \left[1 + \frac{1}{\pi} P.V. \int_a^b \frac{Q'(t)}{t-x}\sqrt{(b-t)(t-a)}dt\right], \quad a < x < b, \tag{1.6}$$

where the above integral is a Cauchy principle value integral.

In view of Definition 1, v_0 is the density of the signed equilibrium on Σ_0. If v_0 happens to be non-negative on Σ_0 then it is the density of the equilibrium measure with external field Q and we are done. If not, then we put

$$d\sigma_0(t) := v_0(t)dt,$$

so that σ_0 is a signed measure of Σ_0. Clearly,

$$U^{\sigma_0}(x) + Q(x) = F_0 \qquad \text{for all } x \in \Sigma_0$$

Let $\sigma_0 = \sigma_0^+ - \sigma_0^-$ be the Jordan decomposition of σ_0 and let

$$\Sigma_1 := \operatorname{supp}(\sigma_0^+).$$

From [15, Lemma 3], $\mu \leq \sigma_0^+$ and $\operatorname{supp}(\mu) \subset \Sigma_1$, so that in determining μ and its support we may restrict ourselves in (1.1) to $\Sigma = \Sigma_1$.

The next step is to determine the signed equilibrium associated with Q on Σ_1, which is to solve the singular equation

$$\int_{\Sigma_1} \log|x-t| d\sigma_1(t) = Q(x) - F_1, \qquad x \in \Sigma_1 \tag{1.7}$$

subject to the condition

$$\int_{\Sigma_1} d\sigma_1 = 1. \tag{1.8}$$

Alternatively, using balayage one derives that the signed measure $\sigma_1 := \sigma_0^+ - \operatorname{Bal}(\sigma_0^-, \Sigma_1)$ satisfies

$$U^{\sigma_1}(x) + Q(x) = F_1 \qquad \text{for all } x \in \Sigma_1,$$

provided Σ_1 is regular (which will be the case in our applications), and by uniqueness it is the solution to (1.7)-(1.8). If Σ_1 is not regular the equality holds q.e. and still uniqueness holds, but we will not elaborate on this generality here.

To describe this process, an operator J was introduced in [15, 6, 7] on all finite signed measures σ on $[a, b]$ with $\int d\sigma = 1$ and $\operatorname{cap}(\operatorname{supp}(\sigma^+)) > 0$ as follows

$$J(\sigma) := \sigma^+ - \operatorname{Bal}(\sigma^-; \operatorname{supp}(\sigma^+)) = \operatorname{Bal}(\sigma; \operatorname{supp}(\sigma^+)).$$

Since the operator J sweeps the negative part of the measure σ onto the support of the positive part, we have that $J(\sigma)^+ \leq \sigma^+$.

The *IBA scheme*

$$\mathcal{I}(\sigma_0) = \{(\Sigma_k, \sigma_k)\}_{k=0}^{\infty}$$

is obtained as the iterates of the operator J applied to a signed (equilibrium) measure σ_0 supported on the set Σ_0, i.e.,

$$\Sigma_k := \operatorname{supp}(\sigma_{k-1}^+), \qquad \sigma_k := J(\sigma_{k-1}) = J^k(\sigma_0), \quad k = 1, 2, \ldots. \tag{1.9}$$

The measures σ_k are signed measures which have a Jordan decomposition $\sigma_k = \sigma_k^+ - \sigma_k^-$. It follows that

$$\sigma_0^+ \geq \sigma_1^+ \geq \cdots \geq \mu, \tag{1.10}$$

and that

$$\Sigma_0 \supset \Sigma_1 \supset \Sigma_2 \supset \cdots \supset S. \tag{1.11}$$

Even for more general Σ_0 one expects from (1.10) and (1.11) that the sequence $\{\sigma_k^+\}_{k=0}^{\infty}$ converges in the weak* topology to the equilibrium measure μ, but this has not been proven yet. If it holds, then we say that the IBA converges. Besides presenting a possible algorithm for numerical calculations, the iterated Balayage algorithm can also be used to prove rigorous results on the support of μ in certain situations. The main difficulty in proving that the iterated Balayage algorithm converges generally, lies in the fact that one has to show that the negative parts σ_k^- tend to zero as k tends to ∞. This can be shown, if one can control the limiting set $\Sigma^* := \cap_{k=1}^{\infty} \Sigma_k$. See for example [15, 6, 7].

Note that the IBA scheme is derived similarly if Σ_0 is union of finitely many arcs of the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, in which case one uses [12, p. 425]. Finally, we remark that recently a continuous version of this algorithm has been used to solve the equilibrium problem for minimal Riesz energy problems for axis-symmetric external fields on the unit sphere $\mathbb{S}^d$ (see [5]).

1.3. Balayage and Kelvin Transform. There is a natural relationship between balayage measures and equilibrium measures. Let us recall that the Kelvin transform $K : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ with center z_0 and radius R is given as the inversion with respect to the circle $\{|z - z_0| = R\}$, namely if $z^* = K(z)$, then z^* lies on the ray stemming from z_0 passing through z and distances satisfy $|z - z_0||z^* - z_0| = R^2$. The distance distortion is given by

$$|z^* - x^*| = \frac{R^2|z - x|}{|z - z_0||x - z_0|}. \tag{1.12}$$

To any measure ν we associate its Kelvin transform $\nu^* = K(\nu)$ as $d\nu^*(x^*) = d\nu(x)$. Observe, that both the point and measure conversion are self-inverse.

Given a compact set A and a point $z_0 \notin A$ we find the balayage $\widehat{\delta_{z_0}}$ of the Dirac-delta measure δ_{z_0} using Riesz's approach [16, Chapter IV, §5], namely if $A^* = K(A)$, then the following relation holds

$$\widehat{\delta_{z_0}} = K(\mu_{A^*}), \tag{1.13}$$

where μ_{A^*} is the equilibrium measure of A^*.

2. Results and examples. Following is our main result.

THEOREM 2. *Let $\Sigma \subseteq \mathbb{R}$ and $w : \Sigma \to [0, \infty)$, $w = \exp(-Q)$ as described in Section 1.1. Suppose that the signed equilibrium on Σ associated with Q exists and is denoted by σ_0. If $\operatorname{supp}(\sigma_0^+)$ consists of at most two intervals A_1, A_2, and σ_0^+ has a concave density on each subinterval, then the equilibrium support $\operatorname{supp}(\mu)$ consists of at most two intervals B_1, B_2, with $B_1 \subset A_1$, $B_2 \subset A_2$, and the equilibrium density is concave on each of these intervals.*

We remark that the theorem remains valid for any $\Sigma \subseteq \mathbb{R}$ compact set as long as σ_0^+ has a support consisting of at most two intervals and σ_0^+ has a concave density. Also, the δ-Hölder continuous first derivative condition on Q must be satisfied only on $\operatorname{supp}(\sigma_0^+)$.

The proof of this theorem relies on a lemma concerning the balayage of a measure onto one or two intervals, which we deem important by itself and include in this section.

LEMMA 3. *Let A be an interval or a union of two intervals, and let ν be a positive measure with compact support in $\mathbb{R}$, such that $\nu(A) = 0$. Then the balayage $\hat{\nu} = \operatorname{Bal}(\nu, A)$ of ν onto A is absolutely continuous with convex density on every subinterval.*

If we consider $\nu = \delta_s$, where δ_s is the Dirac-delta measure with point mass at s, and let $s \to \infty$, then it is well known that densities of the balayage measures $\hat{\delta}_s$ converge to the density of the equilibrium measure μ_A. Therefore, we obtain as a byproduct of our analysis the following corollary.

COROLLARY 4. *The density of the equilibrium measure of the union of any two intervals is convex on every subinterval.*

EXAMPLE 5. Let $\Sigma = [0, 3\pi]$ and $Q(x) := 0.5 \int_0^{3\pi} \ln|x - t| \sin t \, dt$. We can verify that the signed equilibrium is $d\eta = 0.5 \sin t \, dt$. Since it is positive and concave on $[0, \pi] \cup [2\pi, 3\pi]$, we conclude that the equilibrium measure associated with $Q(x)$ is supported on at most two intervals $I_1 \cup I_2$, where $I_1 \subset [0, \pi]$ and $I_2 \subset [2\pi, 3\pi]$, and that the equilibrium density is concave. We remark that by symmetry, in fact the support will consist of two intervals which are symmetric to the point 1.5π. We also remark that this external field is not weak convex in the sense defined in [2, Definition 9], therefore it is an essentially new example in the literature.

EXAMPLE 6. (*Freud weights example*) The following classical Freud weights example $w(x) = e^{-|x|^\lambda}$ (see [19, Example IV.1.15]) provides a nice illustration of our results. For simplicity of the computations, we shall assume that $\lambda = 2$, or equivalently $Q(x) = x^2$, and $\Sigma = \mathbb{R}$. General theory yields that the support of the equilibrium measure associated with $Q(x)$ is compact, so we may restrict ourselves to solving the minimal energy problem on the interval $[-\beta, \beta]$ for some large enough β. Using (1.6) we easily find that the density of the signed equilibrium in this case is given by

$$v_\beta(x) = \frac{2}{\pi}\sqrt{\beta^2 - x^2} - \frac{\beta^2 - 1}{\pi\sqrt{\beta^2 - x^2}}. \tag{2.1}$$

For $\beta > 1$ the function $v_\beta(x)$ is clearly concave, and thus by Theorem 2 the equilibrium support is one interval and the equilibrium measure associated with $Q(x)$ has concave density. Indeed, it is known that $\operatorname{supp}(\mu_Q) = [-1, 1]$ and $d\mu_Q(x) = (2\sqrt{1 - x^2}/\pi)\, dx$. Below we provide an alternative argument to this fact.

Observe that since $v_\beta(1) > 0$ for all $\beta > 1$, the IBA scheme is the collection of nested intervals $\operatorname{supp}(\sigma_k) =: [-\beta_k, \beta_k] \supset [-1, 1]$. It can be calculated that the sequence $\{\beta_k\}$ satisfies the recurrence relation

$$\beta_{k+1} = \sqrt{\frac{\beta_k^2 + 1}{2}}, \quad \beta_0 = \beta.$$

The density of σ_k is simply given by (2.1) with β replaced by β_k. One can easily show that $\beta_k \searrow 1$ as $k \to \infty$, implying that $\operatorname{supp}(\mu_Q) = [-1, 1]$ and $d\mu_Q(x) = (2\sqrt{1 - x^2}/\pi)\, dx$.

The unit circle counterpart of Theorem 2 is not a trivial consequence of the result on the real line and its proof is essentially different, so we formulate it as a separate theorem.

THEOREM 7. *Let $\Sigma \subseteq \mathbb{T}$ and $w : \Sigma \to [0, \infty)$, $w = \exp(-Q)$ as described in Section 1.1. Suppose that the signed equilibrium on Σ associated with Q exists and is denoted by σ_0. If $\operatorname{supp}(\sigma_0^+)$ consists of at most two arcs A_1, A_2, and $d\sigma_0^+ = g(\theta)\, d\theta$ has a concave density $g(\theta)$ on each of the arcs, then the equilibrium support $\operatorname{supp}(\mu)$ consists of at most two arcs B_1, B_2, with $B_1 \subset A_1$, $B_2 \subset A_2$, and the equilibrium density is concave on each of these arcs. Here $d\theta$ indicates the arclength Lebesgue measure on $\mathbb{T}$.*

We remark that the theorem remains valid for any $\Sigma \subseteq \mathbb{T}$ compact set as long as σ_0^+ has a support consisting of at most two arcs and $d\sigma_0^+ = g(\theta)\, d\theta$ has a concave density. Also, the δ-Hölder continuous first derivative condition on Q must be satisfied only on $\operatorname{supp}(\sigma_0^+)$.

The key to the proof of the unit circle case is an analog of Lemma 3.

LEMMA 8. *Let A be an arc or a union of two arcs, and let ν be a positive measure with compact support on $\mathbb{T}$, such that $\nu(A) = 0$. Then the balayage $\hat{\nu} := \operatorname{Bal}(\nu, A)$ of ν onto A is absolutely continuous measure with respect to the Lebesgue arclength measure $d\theta$ and has convex density on every subarc.*

REMARK 9. It is possible to derive Lemma 3 from Lemma 8 through a limiting process, which we illustrate briefly. It is enough to consider only the point mass balayage case. Let $s \in \mathbb{R}$ be fixed, $A = [a, b] \cup [c, d] \subset \mathbb{R}$, and let A_R be the preimage of A under the inversion with center $s + iR$ and radius R. Observe, that A_R consists of two arcs on the circle $|x - s - iR/2| = R/2$, and hence $\operatorname{Bal}(\delta_s, A_R)$ has convex density with respect to the Lebesgue arclength on every subarc. As $R \to \infty$ one can show that the density of $\operatorname{Bal}(\delta_s, A_R)$ approaches that of $\operatorname{Bal}(\delta_s, A)$, which would imply the convexity of the limit. However, instead of following this route, we prefer to prove Lemma 3 directly, as its proof is simpler than that of Lemma 8, as well as illustrative and constructive.

The next theorem establishes another condition which guarantees that the support of the equilibrium measure is an interval. We say that a function $h(x)$ is strictly increasing (decreasing) a.e. on $[-1, 1]$, if there exists a set $G \subset [-1, 1]$ such that G has measure 1 and $h(x)$ is strictly increasing (decreasing) on G.

THEOREM 10. *Let Q be an external field on $[-1, 1]$ as described in Section 1.1. Then the following hold:*

(a) *Let $f(x) := (1 - x^2)Q'(x)$.*
If f is convex on $[-1, 1]$, then the support S is an interval $[b, 1]$, where $b \in [-1, 1)$.
If f is concave on $[-1, 1]$, then the support S is an interval $[-1, b]$, where $b \in (-1, 1]$.

(b) *For $x \in [-1, 1]$ let $g(x) := Q'(x)\sqrt{1 - x^2}$, and let $g(x) = 0$ otherwise. Assume that $g(x)$ is absolutely continuous, $\int_{\mathbb{R}} \int_{\mathbb{R}} |(g'(x + u) - g'(x))/u| du dx < \infty$, and $x \mapsto \int_{\mathbb{R}} (g'(x + u) - g'(x))/u du$ is a continuous function on $(-1, 1)$.*
If $\sqrt{1 - x^2} g'(x)$ is strictly increasing a.e. on $[-1, 1]$, then the support S is an interval $[b, 1]$, where $b \in [-1, 1)$.
If $\sqrt{1 - x^2} g'(x)$ is a strictly decreasing a.e. on $[-1, 1]$, then the support S is an interval $[-1, b]$, where $b \in (-1, 1]$.

EXAMPLE 11. Let

$$Q(x) = c \arcsin(x/a),\ x \in [-1, 1],\ a > 1, c \neq 0.$$

We find that

$$f(x) = c\left[\sqrt{a^2 - x^2} - \frac{a^2 - 1}{\sqrt{a^2 - x^2}}\right].$$

Thus, for $c>0$ the function f is concave and the equilibrium support $S_Q=[-1,b]$. For $c<0$ the function $f(x)$ is convex and $S_Q=[b,1]$. Observe that the external field Q is weakly-convex in the sense of [2], so this is an independent verification of the result there.

The remainder of this paper is devoted to the proofs of our results.

3. Proof of the real line case. We will repeatedly make use of the Chebyshev's Integral Inequality (see [13, p. 1092], [14, pp. 43-44]), which we formulate as a separate lemma and provide a short proof for completeness.

LEMMA 12. *(Chebyshev, 1882) Let f,g,h be integrable functions on $[a,b]$ and let $f\geq 0$ on $[a,b]$.*

(a) Suppose that both g and h are monotone increasing (decreasing). Then

$$\int_a^b (fg)(x)\,dx \int_a^b (fh)(x)\,dx \leq \int_a^b f(x)\,dx \int_a^b (fgh)(x)dx, \tag{3.1}$$

provided all integrals exist.

(b) If h increases and g decreases (h decreases and g increases), then

$$\int_a^b (fg)(x)\,dx \int_a^b (fh)(x)\,dx \geq \int_a^b f(x)\,dx \int_a^b (fgh)(x)dx, \tag{3.2}$$

provided all integrals exist.

Proof. Clearly (b) follows from (a) by substituting g with $(-g)$, so let h,g be both monotone increasing (decreasing). Then

$$f(x)f(y)[g(x)-g(y)][h(x)-h(y)]\geq 0 \ \text{ for any } \ x,y\in[a,b].$$

Integrating the inequality yields

$$\int_a^b\int_a^b f(x)f(y)[g(x)-g(y)][h(x)-h(y)]\,dx\,dy\geq 0,$$

which implies (3.1). □

REMARK 13. In the particular case when $h(x)=x$, Lemma 12 has an interesting geometric interpretation. Suppose f,g, are integrable non-negative functions on $[a,b]$ and g is an increasing function on $[a,b]$. Then (3.1) becomes

$$\frac{\int xf(x)\,dx}{\int f(x)\,dx}\leq\frac{\int xf(x)g(x)\,dx}{\int f(x)g(x)\,dx}. \tag{3.3}$$

Imagine that the $[a,b]$ interval is a wire with density $f(x)$. The above inequality says that the center of mass will move to the right if we multiply the density by an increasing non-negative function $g(x)$.

We now continue with the proof of Lemma 3.

Proof of Lemma 3. It is enough to prove the lemma only for Dirac delta measures δ_s, $s\notin A$, because in general we have the representation

$$\frac{d\hat{\nu}}{dt}=\int_{\mathrm{supp}(\nu)}\frac{d\hat{\delta}_s}{dt}\,d\nu(s)=:\int_{\mathrm{supp}(\nu)}f(t,s)\,d\nu(s), \tag{3.4}$$

where $f(t,s):=d\hat{\delta}_s/dt$. Then the convexity of $\hat{\nu}$ can be easily derived from that of $f(t,s)$ by integrating the inequality

$$f(\alpha t+(1-\alpha)y,s)\leq\alpha f(t,s)+(1-\alpha)f(y,s),\quad 0\leq\alpha\leq 1,\quad t,y\in A$$

with respect to $d\nu(s)$ and using (3.4) (recall that $\nu(A)=0$).

We now consider two cases.
Case 1: The set A consists of one interval, for simplicity $A = [-1, 1]$.
From [19, Chapter II, Corollary 4.12] we have that

$$f(t, s) = \frac{1}{\pi} \frac{\sqrt{s^2 - 1}}{|s - t|\sqrt{1 - t^2}} \tag{3.5}$$

We claim that $f_{tt}(t, s) > 0$ for all $t \in (-1, 1)$. Without loss of generality let $s > 1$. Then we find that

$$f_{tt}(t, s) = \frac{\sqrt{s^2 - 1}}{\pi} \left(\frac{2}{(s - t)^3\sqrt{1 - t^2}} + \frac{s + t}{(s - t)^2(1 - t^2)^{3/2}} + \frac{3t^2}{(s - t)(1 - t^2)^{5/2}} \right),$$

which verifies the claim, since $s + t > 0$ and $s - t > 0$.

Case 2: The set A consists of two intervals, i.e., $A = [a, b] \cup [c, d]$. Assume first that $s \in (b, c)$. Applying Kelvin transform K with center s and radius 1 we obtain $A^* := K(A) = [b^*, a^*] \cup [d^*, c^*]$ (see Section 1.3).

The density of the equilibrium measure of A^* (see [21, Lemma 4.4.1]) is given by

$$\frac{d\mu_{A^*}}{dt^*} = \frac{|t^* - y^*|}{\pi\sqrt{|t^* - a^*||t^* - b^*||t^* - c^*||t^* - d^*|}}, \tag{3.6}$$

where $y^* \in (a^*, d^*)$ is determined from the equation

$$\int_{a^*}^{d^*} \frac{t^* - y^*}{\pi\sqrt{|t^* - a^*||t^* - b^*||t^* - c^*||t^* - d^*|}}\, dt^* = 0. \tag{3.7}$$

Using the distance distortion formula (1.12) and the balayage/equilibrium relation (1.13) we derive the balayage density as

$$\frac{d\hat{\delta}_s}{dt} = f(t, s) := \frac{\sqrt{|s - a||s - b||s - c||s - d|}}{\pi|s - y|} \frac{|t - y|}{|t - s|\sqrt{|t - a||t - b||t - c||t - d|}}, \quad t \in A. \tag{3.8}$$

Let

$$\phi(t) := \log(f(t, s)) = \log\left(\frac{|t - y|}{|t - s|\sqrt{|t - a||t - b||t - c||t - d|}} \right) + c(s).$$

We shall prove that for fixed $s \in (b, c)$, we have $\phi''(x) > 0$ for all $x \in (a, b) \cup (c, d)$. This amounts to the inequality

$$\frac{1}{(x - y)^2} < \frac{1}{(x - s)^2} + \frac{1}{2}\left\{ \frac{1}{(x - a)^2} + \frac{1}{(x - b)^2} + \frac{1}{(x - c)^2} + \frac{1}{(x - d)^2} \right\}, \text{ for all } x \in A. \tag{3.9}$$

(Observe that if $y^* = s$, then $y = \infty$ and the factors $|t - y|$ and $|s - y|$ in (3.8) are omitted and (3.9) obviously holds.)

If $y^* \in (a^*, d^*) \setminus \{s\}$, then $y \in (-\infty, a) \cup (d, \infty)$. Without loss of generality we may assume $y \in (-\infty, a)$. Clearly, for $x \in [c, d]$ (3.9) holds. So, let us fix $x \in [a, b]$. From (3.7) we derive that

$$y^* - a^* = \frac{\displaystyle\int_{a^*}^{d^*} \frac{t^* - a^*}{\sqrt{|t^* - a^*||t^* - b^*||t^* - c^*||t^* - d^*|}}\, dt^*}{\displaystyle\int_{a^*}^{d^*} \frac{1}{\sqrt{|t^* - a^*||t^* - b^*||t^* - c^*||t^* - d^*|}}\, dt^*} \tag{3.10}$$

Since $\sqrt{|t^*-c^*|}$ is decreasing on $[a^*,d^*]$ we can apply Lemma 12 (see also Remark 13) to estimate that

$$y^*-a^* \geq \frac{\displaystyle\int_{a^*}^{d^*} \frac{t^*-a^*}{\sqrt{|t^*-a^*||t^*-b^*||t^*-d^*|}}\,dt^*}{\displaystyle\int_{a^*}^{d^*} \frac{1}{\sqrt{|t^*-a^*||t^*-b^*||t^*-d^*|}}\,dt^*} \geq \frac{\sqrt{a^*-b^*}}{\sqrt{d^*-b^*}} \cdot \frac{\displaystyle\int_{a^*}^{d^*} \sqrt{\frac{t^*-a^*}{d^*-t^*}}\,dt^*}{\displaystyle\int_{a^*}^{d^*} \frac{1}{\sqrt{(t^*-a^*)(d^*-t^*)}}\,dt^*} \tag{3.11}$$

The second fraction evaluates to $(d^*-a^*)/2$, thus reducing (3.11) to

$$y^*-a^* \geq \frac{\sqrt{|a^*-b^*|}}{\sqrt{|d^*-b^*|}}\frac{d^*-a^*}{2}, \tag{3.12}$$

which after the Kelvin transformation becomes

$$\frac{|y-a|}{|y-s|} \geq \frac{\sqrt{|d-s||a-b|}}{\sqrt{|a-s||d-b|}}\frac{|d-a|}{2|d-s|}. \tag{3.13}$$

Rewriting (3.13) yields

$$\frac{1}{\sqrt{|y-a|}} \leq \frac{2\sqrt{|y-a||a-s|}}{|y-s|} \cdot \frac{\sqrt{|d-s||d-b|}}{|d-a|} \cdot \frac{1}{\sqrt{|a-b|}} \leq \frac{1}{\sqrt{|a-b|}}, \tag{3.14}$$

where each of the two fractions above is less than 1.

On the other hand, one easily derives that

$$\min_{x\in[a,b]} \frac{1}{2}\left\{\frac{1}{(a-x)^2}+\frac{1}{(b-x)^2}\right\} = \frac{4}{(b-a)^2}.$$

So, the inequality (3.14) implies (3.9) at least with a factor of four in excess.

$$\frac{1}{(y-x)^2} \leq \frac{1}{(y-a)^2} \leq \frac{1}{(a-b)^2} \leq \frac{1}{8}\left\{\frac{1}{(a-x)^2}+\frac{1}{(b-x)^2}\right\}.$$

This establishes the lemma when $s\in(b,c)$, and hence for any measure ν with $\mathrm{supp}(\nu)\subset[b,c]$ with $\nu(A)=0$. If we assume that $s\in(-\infty,a)\cup(d,\infty)$, then the balayage $\mu=\mathrm{Bal}(\delta_s,[a,d])$ has a convex density by Case 1. However, from the properties of balayage measure we have that

$$\mathrm{Bal}(\delta_s,[a,b]\cup[c,d]) = \mathrm{Bal}(\mu,[a,b]\cup[c,d]) = \mu_{|_A} + \mathrm{Bal}(\mu_{|_{[b,c]}},A).$$

Both measures on the right have convex densities on A, the first, as a restriction of a measure with a convex density on the entire interval $[a,d]$, and the second by what we just proved. Hence, $\hat{\delta}_s$ has convex density and the lemma is proved. □

We are now ready for the

Proof of Theorem 2. The proof is based on the iterated balayage algorithm discussed in Section 1.2. Let $\Sigma_0 := \Sigma$ and σ_0 denote the signed equilibrium associated with Q. By the assumption of the theorem the latter exists, and if $\sigma_0 = \sigma_0^+ - \sigma_0^-$ is the Jordan decomposition of σ_0, the measure σ_0^+ has concave density and its support $\Sigma_1 := \mathrm{supp}(\sigma_0^+)$ consists of at most two intervals. Then from Lemma 3, $\mathrm{Bal}(\sigma_0^-,\Sigma_1)$ is convex and thus, $\sigma_1 = J(\sigma_0) = \sigma_0^+ - \mathrm{Bal}(\sigma_0^-,\Sigma_1)$ has concave density on Σ_1. Therefore, $\Sigma_2 := \mathrm{supp}(\sigma_1^+)$ consists of at most two intervals (nested in at most two intervals that make up Σ_1).

Continuing, in this way we obtain an IBA sequence of nested compact sets

$$\Sigma_0 \supset \Sigma_1 \supset \dots \Sigma_n \supset \dots \supset \mathrm{supp}(\mu_Q)\,,$$

each made of at most two intervals.

Now we show that the density of σ_n, which we will denote by $v_n(x)$, is converging to the density of the equilibrium measure μ_Q associated with the external field $Q(x)$.

Let us assume that for all n we have $\Sigma_n = [a_n, b_n] \cup [c_n, d_n]$, where $[a_n, b_n]$ and $[c_n, d_n]$ are two non-trivial disjoint intervals. If it was not the case, the proof would be the same. (We remark that even if Σ_0 consisted of two intervals, we may "lose" one in the IBA if at one step $v_n \leq 0$ on $[a_n, b_n]$ or on $[c_n, d_n]$.)

Let $\lim a_n = a, \lim b_n = b, \lim c_n = c, \lim d_n = d$. We have

$$\begin{aligned} Q(x) - F_n &= \int_{\Sigma_n} \log|t-x| v_n(t)\,dt \\ &= \left(\int_{\Sigma_{n+1}} + \int_{\Sigma_n \setminus \Sigma_{n+1}}\right) \log|t-x| v_n(t) dt \\ &= \left(\int_{[a_{n+1},b_{n+1}]\cup[c_{n+1},d_{n+1}]} + \int_{[a_n,a_{n+1}]\cup[b_{n+1},b_n]\cup[c_n,c_{n+1}]\cup[d_{n+1},d_n]}\right) \log|t-x| v_n(t) dt. \end{aligned} \tag{3.15}$$

We claim that

$$\int_{[a_n,a_{n+1}]\cup[b_{n+1},b_n]\cup[c_n,c_{n+1}]\cup[d_{n+1},d_n]} v_n(t) dt \to 0. \tag{3.16}$$

If not, then we can chose a subsequence, denoted by n for simplicity, and $m_i \leq 0, i = 1, \ldots, 4$, such that

$$\int_{[a_n,a_{n+1}]} v_n(t)dt \to m_1, \int_{[b_{n+1},b_n]} v_n(t)dt \to m_2, \int_{[c_n,c_{n+1}]} v_n(t)dt \to m_3, \int_{[d_{n+1},d_n]} v_n(t)dt \to m_4, \sum m_i < 0.$$

Since $v_n\Big|_{[a,b]\cup[c,d]}$ is a bounded decreasing sequence, letting $v := \lim v_n$, we can use the dominated convergence theorem and mean value theorem to derive that for any $x \in (a, b) \cup (c, d)$

$$\lim(Q(x) - F_n) = \int_{[a,b]\cup[c,d]} \log|t-x| v(t) dt + m_1 \log|a-x| + m_2 \log|b-x| + m_3 \log|c-x| + m_4 \log|d-x|. \tag{3.17}$$

This shows that F_n has a finite limit. We also see that we must have $m_1 = m_2 = m_3 = m_4 = 0$. For example, if $m_1 < 0$, let $x = a$ in (3.15) and let $n \to \infty$. The right-hand side of (3.15) is approaching positive infinity while the left hand-side has finite limit, which is a contradiction.

Let $F := \lim F_n$. From (3.15) we get that for any $x \in (a, b) \cup (c, d)$

$$Q(x) - F = \int_{[a,b]\cup[c,d]} \log|t-x| v(t) dt.$$

From (3.16) it is also clear that $v(t)$ is a probability density function. The equilibrium measure μ_Q minimizes the weighted energy on Σ and therefore on $[a, b] \cup [c, d]$, too. Also, the support of μ_Q is a subset of $[a, b] \cup [c, d]$. It follows that $v(x)$ is the density of the equilibrium measure (see [19, Theorem I.3.3]).

If $a = b$ then $\operatorname{supp}(\mu_Q) = [c, d]$. If $c = d$ then $\operatorname{supp}(\mu_Q) = [a, b]$. Finally, if $a < b$ and $c < d$ then $\operatorname{supp}(\mu_Q) = [a, b] \cup [c, d]$. □

4. Proof of the unit circle case. We proceed first with the proof of Lemma 8.

Proof of Lemma 8. As in the proof of Lemma 3 above, it is enough to verify the Lemma for case of a point mass, so without loss of generality we assume that $\nu = \delta_s$, where $s \in \mathbb{T} \setminus A$.

Given two points on the unit circle $e^{i\phi}$ and $e^{i\theta}$ we denote by $[e^{i\phi}, e^{i\theta}]$ the closed arc that connects the points counterclockwise. In this notation, its complement relative to the unit circle would be the open arc $(e^{i\theta}, e^{i\phi})$. Since the case of an arc easily follows from the case of two arcs as we deform one of the arcs to a point, we set $A =$

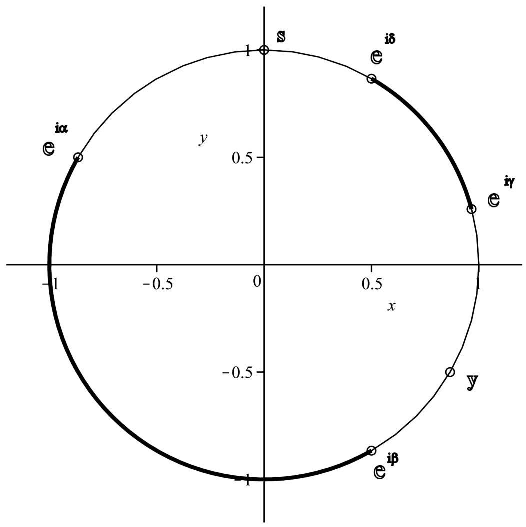

FIG. 4.1. *Circle case*

$[e^{i\alpha}, e^{i\beta}] \cup [e^{i\gamma}, e^{i\delta}]$, where $e^{i\alpha}, e^{i\beta}, e^{i\gamma}, e^{i\delta}$ are points on the unit circle ordered counterclockwise. Without loss of generality we assume that $s = i \in (e^{i\delta}, e^{i\alpha})$ (see Fig. 4.1). For simplicity assume also that all angles below are given in the interval $[\pi/2, 5\pi/2)$, meaning in particular that $\pi/2 < \alpha < \beta < \gamma < \delta < 5\pi/2$.

To find the balayage $\widehat{\nu}$, we observe that after a Kelvin transform centered at s with a radius $\sqrt{2}$ the unit circle is sent to the real line and n $A^* = [a, b] \cup [c, d]$, where $a = \left(e^{i\alpha}\right)^*$, $b = \left(e^{i\beta}\right)^*$, $c = \left(e^{i\gamma}\right)^*$, and $d = \left(e^{i\delta}\right)^*$ (see Fig. 4.2). Using Riesz's approach (Section 1.3) we find $\widehat{\nu} = K(\mu_{A^*})$. Recall that (see (3.6))

$$d\mu_{A^*} = \frac{|t^* - y^*|}{\pi\sqrt{|t^* - a||t^* - b||t^* - c||t^* - d|}}\, dt^*, \quad t^* \in [a, b] \cup [c, d], \tag{4.1}$$

where $t^* = \left(e^{i\theta}\right)^* \in A$ and $y^* \in (b, c)$ is determined from the equation

$$\int_b^c \frac{x^* - y^*}{\pi\sqrt{|x^* - a||x^* - b||x^* - c||x^* - d|}}\, dx^* = 0. \tag{4.2}$$

Since the relationship between the Lebesgue measures on $\mathbb{R}$ and $\mathbb{T}$ is given by

$$\frac{dt^*}{|t^* - s|} = \frac{|d\,t|}{|t - s|} = \frac{d\theta}{|e^{i\theta} - s|},$$

we find that

$$dt = \frac{2d\theta}{|e^{i\theta} - s|^2}. \tag{4.3}$$

Let $y := e^{i\phi} := (y^*)^*$ with $\phi \in (\beta, \gamma)$. Using (1.12) and the formula $|e^{i\xi} - e^{i\zeta}| = 2|\sin\left(\frac{\xi - \zeta}{2}\right)|$, we conclude that (recall the order of the angles)

$$d\widehat{\nu} = C\frac{|\sin\left(\frac{\theta - \phi}{2}\right)|\, d\theta}{\sin\left(\frac{\theta - \pi/2}{2}\right)\sqrt{|\sin\left(\frac{\theta - \alpha}{2}\right)\sin\left(\frac{\theta - \beta}{2}\right)\sin\left(\frac{\theta - \gamma}{2}\right)\sin\left(\frac{\theta - \delta}{2}\right)|}}, \quad \theta \in [\alpha, \beta] \cup [\gamma, \delta], \tag{4.4}$$

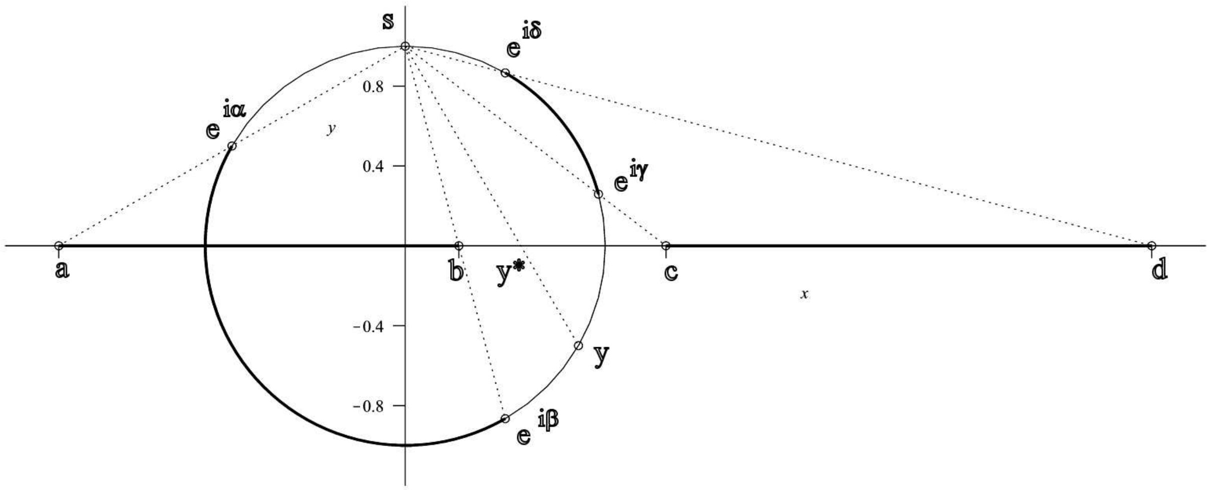

FIG. 4.2. *Kelvin transformation - circle case*

where

$$C = \frac{\sqrt{\sin\left(\frac{\alpha-\pi/2}{2}\right)\sin\left(\frac{\beta-\pi/2}{2}\right)\sin\left(\frac{\gamma-\pi/2}{2}\right)\sin\left(\frac{\delta-\pi/2}{2}\right)}}{2\pi\sin\left(\frac{\phi-\pi/2}{2}\right)}.$$

If $x^* = \left(e^{i\psi}\right)^*$, then

$$x^* - y^* = \frac{\sin\left(\frac{\psi-\phi}{2}\right)}{2\sin\left(\frac{\phi-\pi/2}{2}\right)\sin\left(\frac{\psi-\pi/2}{2}\right)}, \quad x^*, y^* \in [b, c].$$

Indeed, the equality holds with absolute values by (1.12) and since the signs of $x^* - y^*$ and $\sin\left(\frac{\psi-\phi}{2}\right)$ are the same we can remove the absolute value. Therefore, (4.2) implies that ϕ is determined uniquely by

$$\int_\beta^\gamma \frac{\sin\left(\frac{\psi-\phi}{2}\right)}{\sin\left(\frac{\psi-\pi/2}{2}\right)\sqrt{\left|\sin\left(\frac{\psi-\alpha}{2}\right)\sin\left(\frac{\psi-\beta}{2}\right)\sin\left(\frac{\psi-\gamma}{2}\right)\sin\left(\frac{\psi-\delta}{2}\right)\right|}}\,d\psi = 0, \tag{4.5}$$

which is equivalent to

$$\cot\left(\frac{\phi-\theta}{2}\right) = \frac{\displaystyle\int_\beta^\gamma \cot\left(\frac{\psi-\theta}{2}\right)\frac{\sin\left(\frac{\psi-\theta}{2}\right)}{\sin\left(\frac{\psi-\pi/2}{2}\right)\sqrt{\left|\sin\left(\frac{\psi-\alpha}{2}\right)\sin\left(\frac{\psi-\beta}{2}\right)\sin\left(\frac{\psi-\gamma}{2}\right)\sin\left(\frac{\psi-\delta}{2}\right)\right|}}\,d\psi}{\displaystyle\int_\beta^\gamma \frac{\sin\left(\frac{\psi-\theta}{2}\right)}{\sin\left(\frac{\psi-\pi/2}{2}\right)\sqrt{\left|\sin\left(\frac{\psi-\alpha}{2}\right)\sin\left(\frac{\psi-\beta}{2}\right)\sin\left(\frac{\psi-\gamma}{2}\right)\sin\left(\frac{\psi-\delta}{2}\right)\right|}}\,d\psi} \tag{4.6}$$

for any convenient choice of θ.

Recall that our goal is to show that for any $\pi/2 < \alpha < \beta < \gamma < \delta < 5\pi/2$, the balayage density function

$$N(\theta) := \frac{|\sin\left(\frac{\theta-\phi}{2}\right)|}{\sin\left(\frac{\theta-\pi/2}{2}\right)\sqrt{|\sin\left(\frac{\theta-\alpha}{2}\right)\sin\left(\frac{\theta-\beta}{2}\right)\sin\left(\frac{\theta-\gamma}{2}\right)\sin\left(\frac{\theta-\delta}{2}\right)|}}, \quad \theta\in[\alpha,\beta]\cup[\gamma,\delta]$$

is convex, provided ϕ satisfies (4.6). We will actually show more, namely that $(\log(N(\theta)))'' > 0$. The latter is equivalent to verifying for all $\theta \in [\alpha,\beta]\cup[\gamma,\delta]$ the inequality

$$\csc^2\left(\frac{\theta-\phi}{2}\right) \le \csc^2\left(\frac{\theta-\pi/2}{2}\right) + \frac{1}{2}\left[\csc^2\left(\frac{\theta-\alpha}{2}\right) + \csc^2\left(\frac{\theta-\beta}{2}\right) + \csc^2\left(\frac{\theta-\gamma}{2}\right) + \csc^2\left(\frac{\theta-\delta}{2}\right)\right]. \tag{4.7}$$

Let us fix $\theta \in [\alpha,\beta]$ (the case $\theta\in[\gamma,\delta]$ is considered similarly). If $\phi-\theta \ge \pi$, then

$$\csc^2\left(\frac{\theta-\phi}{2}\right) = \frac{4}{|e^{i\phi}-e^{i\theta}|} < \frac{4}{|e^{i\pi/2}-e^{i\theta}|} = \csc^2\left(\frac{\theta-\pi/2}{2}\right)$$

and (4.7) holds trivially. Henceforth, assume that $\phi - \theta < \pi$.

We now perform several perturbations to simplify the problem. First, we study the effect of substituting δ with $5\pi/2$. Let $\phi_1 \in (\beta,\gamma)$ be uniquely determined by

$$\cot\left(\frac{\phi_1-\theta}{2}\right) = \frac{\displaystyle\int_\beta^\gamma \cot\left(\frac{\psi-\theta}{2}\right)\frac{\sin\left(\frac{\psi-\theta}{2}\right)}{\sin\left(\frac{\psi-\pi/2}{2}\right)\sqrt{|\sin\left(\frac{\psi-\alpha}{2}\right)\sin\left(\frac{\psi-\beta}{2}\right)\sin\left(\frac{\psi-\gamma}{2}\right)\sin\left(\frac{\psi-5\pi/2}{2}\right)|}}\,d\psi}{\displaystyle\int_\beta^\gamma \frac{\sin\left(\frac{\psi-\theta}{2}\right)}{\sin\left(\frac{\psi-\pi/2}{2}\right)\sqrt{|\sin\left(\frac{\psi-\alpha}{2}\right)\sin\left(\frac{\psi-\beta}{2}\right)\sin\left(\frac{\psi-\gamma}{2}\right)\sin\left(\frac{\psi-5\pi/2}{2}\right)|}}\,d\psi}. \tag{4.8}$$

Observe that the monotonicity of $\cot((\psi-\theta)/2)$ guarantees the existence and uniqueness of ϕ_1. Applying Lemma 12 with

$$h(\psi) = \cot\left(\frac{\psi-\theta}{2}\right), \quad g(\psi) = \sqrt{\frac{\sin\left(\frac{\delta-\psi}{2}\right)}{\sin\left(\frac{5\pi/2-\psi}{2}\right)}} = \sqrt{\cos\left(\frac{5\pi/2-\delta}{2}\right) - \cot\left(\frac{5\pi/2-\psi}{2}\right)\sin\left(\frac{5\pi/2-\delta}{2}\right)}$$

and using that both $h(\psi)$ and $g(\psi)$ are monotone decreasing functions on (β,γ), we conclude that $\cot\left(\dfrac{\phi-\theta}{2}\right) < \cot\left(\dfrac{\phi_1-\theta}{2}\right)$, or that $\phi_1 < \phi$.

Similarly, if $\phi_2 \in (\beta,\gamma)$ denotes the unique solution obtained when $\alpha = \theta$ in (4.8), we conclude that $\phi_2 < \phi_1$. Indeed, if

$$\cot\left(\frac{\phi_2-\theta}{2}\right) = \frac{\displaystyle\int_\beta^\gamma \cot\left(\frac{\psi-\theta}{2}\right)\frac{\sqrt{\sin\left(\frac{\psi-\theta}{2}\right)}}{\left(\sin\left(\frac{\psi-\pi/2}{2}\right)\right)^{3/2}\sqrt{\sin\left(\frac{\psi-\beta}{2}\right)\sin\left(\frac{\gamma-\psi}{2}\right)}}\,d\psi}{\displaystyle\int_\beta^\gamma \frac{\sqrt{\sin\left(\frac{\psi-\theta}{2}\right)}}{\left(\sin\left(\frac{\psi-\pi/2}{2}\right)\right)^{3/2}\sqrt{\sin\left(\frac{\psi-\beta}{2}\right)\sin\left(\frac{\gamma-\psi}{2}\right)}}\,d\psi}, \tag{4.9}$$

we again apply Lemma 12, this time with

$$g(\psi)=\sqrt{\frac{\sin\left(\frac{\psi-\alpha}{2}\right)}{\sin\left(\frac{\psi-\theta}{2}\right)}}=\sqrt{\cos\left(\frac{\theta-\alpha}{2}\right)+\cot\left(\frac{\psi-\theta}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)},$$

which is still decreasing function for $\psi\in(\beta,\gamma)$.

Finally, let $\phi_3\in(\beta,\gamma)$ be derived from (4.9) with θ instead of $\pi/2$. Then we have

$$\cot\left(\frac{\phi_3-\theta}{2}\right)=\frac{\displaystyle\int_\beta^\gamma \cot\left(\frac{\psi-\theta}{2}\right)\frac{1}{\sin\left(\frac{\psi-\theta}{2}\right)\sqrt{\sin\left(\frac{\psi-\beta}{2}\right)\sin\left(\frac{\gamma-\psi}{2}\right)}}\,d\psi}{\displaystyle\int_\beta^\gamma \frac{1}{\sin\left(\frac{\psi-\theta}{2}\right)\sqrt{\sin\left(\frac{\psi-\beta}{2}\right)\sin\left(\frac{\gamma-\psi}{2}\right)}}\,d\psi}. \tag{4.10}$$

Applying Lemma 12 with the decreasing function

$$g(\psi)=\left(\frac{\sin\left(\frac{\psi-\pi/2}{2}\right)}{\sin\left(\frac{\psi-\theta}{2}\right)}\right)^{3/2}=\left(\cos\left(\frac{\theta-\pi/2}{2}\right)+\cot\left(\frac{\psi-\theta}{2}\right)\sin\left(\frac{\theta-\pi/2}{2}\right)\right)^{3/2}$$

we obtain $\phi_3<\phi_2$. In Lemma 14 below we will show for the so derived ϕ_3 the inequality

$$\csc^2\left(\frac{\phi_3-\theta}{2}\right)\le 2+\frac{1}{2}\left[\csc^2\left(\frac{\theta-\beta}{2}\right)+\csc^2\left(\frac{\theta-\gamma}{2}\right)\right], \tag{4.11}$$

which coupled with $\theta<\beta<\phi_3<\phi<\gamma$ and $\phi-\theta<\pi$ proves (4.7) so the proof of Lemma 8 will be complete. $\square$

We now prove that (4.11) holds provided ϕ_3 satisfies (4.10). Since we find the inequality interesting in its own right we formulate it as a separate lemma. Without loss of generality we could assume that $\beta=0<\gamma<\theta<2\pi$.

LEMMA 14. *Let* $0<\gamma<\theta<2\pi$. *Then*

$$\left(\frac{\displaystyle\int_0^\gamma \frac{\cot\left(\frac{\theta-t}{2}\right)dt}{\sin\left(\frac{\theta-t}{2}\right)\sqrt{\sin\left(\frac{t}{2}\right)\sin\left(\frac{\gamma}{2}-\frac{t}{2}\right)}}}{\displaystyle\int_0^\gamma \frac{dt}{\sin\left(\frac{\theta-t}{2}\right)\sqrt{\sin\left(\frac{t}{2}\right)\sin\left(\frac{\gamma}{2}-\frac{t}{2}\right)}}}\right)^2\le 1+\frac{1}{2}\left[\frac{1}{\sin^2\left(\frac{\theta}{2}\right)}+\frac{1}{\sin^2\left(\frac{\theta-\gamma}{2}\right)}\right]. \tag{4.12}$$

Proof. First we compute:

$$I_1=\int_0^\gamma \frac{dt}{\sin\left(\frac{\theta-t}{2}\right)\sqrt{\sin\left(\frac{t}{2}\right)\sin\left(\frac{\gamma}{2}-\frac{t}{2}\right)}} \tag{4.13}$$

Making a substitution $y=(\cos(t-\gamma/2)-\cos(\gamma/2))/(1-\cos(\gamma/2))$ in (4.13) leads to

$$\begin{aligned}I_1&=\frac{4\sin\left(\frac{\theta-\gamma/2}{2}\right)}{1-\cos(\frac{\gamma}{2})}\int_0^1\frac{1}{y+c}\frac{dy}{\sqrt{y(1-y)}}=\frac{4\sin\left(\frac{\theta-\gamma/2}{2}\right)}{1-\cos(\frac{\gamma}{2})}\frac{\pi}{\sqrt{c(c+1)}}\\&=\frac{2\sqrt{2}\pi}{\sqrt{\cos(\gamma/2)-\cos(\theta-\gamma/2)}},\end{aligned} \tag{4.14}$$

where $c = c(\theta,\gamma) := \dfrac{\cos(\gamma/2) - \cos(\theta-\gamma/2)}{1-\cos(\gamma/2)} > 0$.

Next, we compute similarly,

$$\begin{aligned} I_2 &:= \int_0^\gamma \frac{\cos((\theta-t)/2)}{\sin^2((\theta-t)/2)} \frac{dt}{\sqrt{\sin(t/2)\sin(\gamma/2-t/2)}} \\ &= \sqrt{2}\int_0^{\gamma/2} \left[\frac{\cos\left(\frac{\theta-\gamma/2}{2} - u/2\right)}{\sin^2\left(\frac{\theta-\gamma/2}{2} - u/2\right)} + \frac{\cos\left(\frac{\theta-\gamma/2}{2} + y/2\right)}{\sin^2\left(\frac{\theta-\gamma/2}{2} + u/2\right)} \right] \frac{du}{\sqrt{\cos(u) - \cos(\gamma/2)}}. \end{aligned}$$

Using the identity

$$\frac{\cos(\alpha-\alpha')}{\sin^2(\alpha-\alpha')} + \frac{\cos(\alpha+\alpha')}{\sin^2(\alpha+\alpha')} = \frac{8\cos(\alpha)\cos(\alpha')(1-1/2(\cos(2\alpha)+\cos(2\alpha')))}{(\cos(2\alpha')-\cos(2\alpha))^2}$$

that holds for any real α and α', and the substitution $y = (\cos u - \cos(\gamma/2))/(1-\cos(\gamma/2))$, we get

$$I_2 = \frac{4\cos((\theta-\gamma/2))}{1-\cos(\gamma/2)} \left[\frac{2(1-\cos(\theta-\gamma/2))}{1-\cos(\gamma/2)} \int_0^1 \frac{1}{(y+c)^2}\frac{dy}{\sqrt{y(1-y)}} - \int_0^1 \frac{1}{y+c}\frac{dy}{\sqrt{y(1-y)}} \right]. \tag{4.15}$$

Since

$$\int_0^1 \frac{1}{(y+c)^2}\frac{dy}{\sqrt{y(1-y)}} = -\frac{d}{dc}\left(\int_0^1 \frac{1}{y+c}\frac{dy}{\sqrt{y(1-y)}} \right) = -\frac{d}{dc}\frac{\pi}{\sqrt{c(1+c)}},$$

we obtain

$$\int_0^1 \frac{dy}{(y+c)^2\sqrt{y(1-y)}} = \frac{\pi(2c+1)}{2\sqrt{c(c+1)}c(c+1)}.$$

Substituting in (4.15) we find that

$$I_2 = \frac{2\sqrt{2}\sin(\theta-\gamma/2)\pi}{(\cos(\gamma/2)-\cos(\theta-\gamma/2))^{3/2}}. \tag{4.16}$$

Thus, we see from (4.14) and (4.16) that

$$\begin{aligned} (I_2/I_1)^2 &= (1/4)(\cot(\theta/2) + \cot((\theta-\gamma)/2))^2 \leq (1/2)(\cot^2(\theta/2) + \cot^2((\theta-\gamma)/2))) \\ &< (1/2)(\csc^2(\theta/2) + \csc^2((\theta-\gamma)/2)) < 1 + 1/2(\csc^2(\theta/2) + \csc^2((\theta-\gamma)/2)), \end{aligned}$$

which concludes the proof. □

Proof of Theorem 7. The proof of this theorem follows word for word the argument in Theorem 2, where instead of Lemma 3 we use Lemma 8. □

Proof of Theorem 10. From (1.6) with $[a,b] = [-1,1]$ we can write

$$v(x) = \frac{1}{\pi\sqrt{1-x^2}} \left[1 + \frac{1}{\pi}\int_{-1}^1 \frac{f(t)-f(x)}{(t-x)\sqrt{1-t^2}}\,dt \right]. \tag{4.17}$$

Here we used the fact that

$$P.V. \int_{-1}^1 \frac{1}{(t-x)\sqrt{1-t^2}}\,dt = 0,$$

which follows from differentiation of the equilibrium potential on $[-1,1]$

$$\int_{-1}^{1} \log \frac{1}{|x-t|} \frac{dt}{\pi\sqrt{1-t^2}} = \log 2, \quad x \in [-1,1].$$

Proof of part a). If f is identically zero, then Q is constant and $S=[-1,1]$. Note that f cannot be a linear function (unless $f \equiv 0$) because of the δ-Hölder continuity assumption on Q'. So let f be convex but not a linear function. Let $g_t(x) = (f(t)-f(x))/(t-x)$, and $x_1 < x_2$. Then $g_t(x_1) \le g_t(x_2)$ for all $t \in [-1,1]$ and $g_t(x_1) < g_t(x_2)$ holds on a set of positive measure. Integrating over $[-1,1]$ with respect to t, we thus derive that $\sqrt{1-x^2}v(x)$ is strictly increasing. The claim now follows from [15, Theorem 2]. The proof is similar when f is concave.

Proof of part b). Let $\sqrt{1-x^2}g'(x)$ be strictly increasing (strictly decreasing) a.e. on $[-1,1]$. Note that $\sqrt{1-x^2}v(x)$ is strictly increasing (strictly decreasing) function a.e. if the following is positive (negative):

$$\frac{d}{dx}\int_{-1}^{1} \frac{\sqrt{1-t^2}Q'(t)}{t-x}dt = \int_{-1}^{1} \frac{(\sqrt{1-t^2}Q'(t))'}{t-x}dt = \int_{-1}^{1} \frac{\sqrt{1-t^2}(\sqrt{1-t^2}Q'(t))' - \sqrt{1-x^2}(\sqrt{1-x^2}Q'(x))'}{(t-x)\sqrt{1-t^2}}dt.$$

We used that

$$\frac{d}{dx}\int_{-1}^{1} \frac{g(t)}{t-x}dt = \frac{d}{dx}\int_{\mathbb{R}} \frac{g(x+u)-g(x)}{u}du = \int_{\mathbb{R}} \frac{g'(x+u)-g'(x)}{u}du = \int_{-1}^{1} \frac{g'(t)}{t-x}du.$$

We could differentiate inside the parametric integral because of [1, Lemma 13].

The claim of part b) now follows from [15, Theorem 2]. ∎

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