Robust Reduced Order Modeling for Complex Multi-scale Problems

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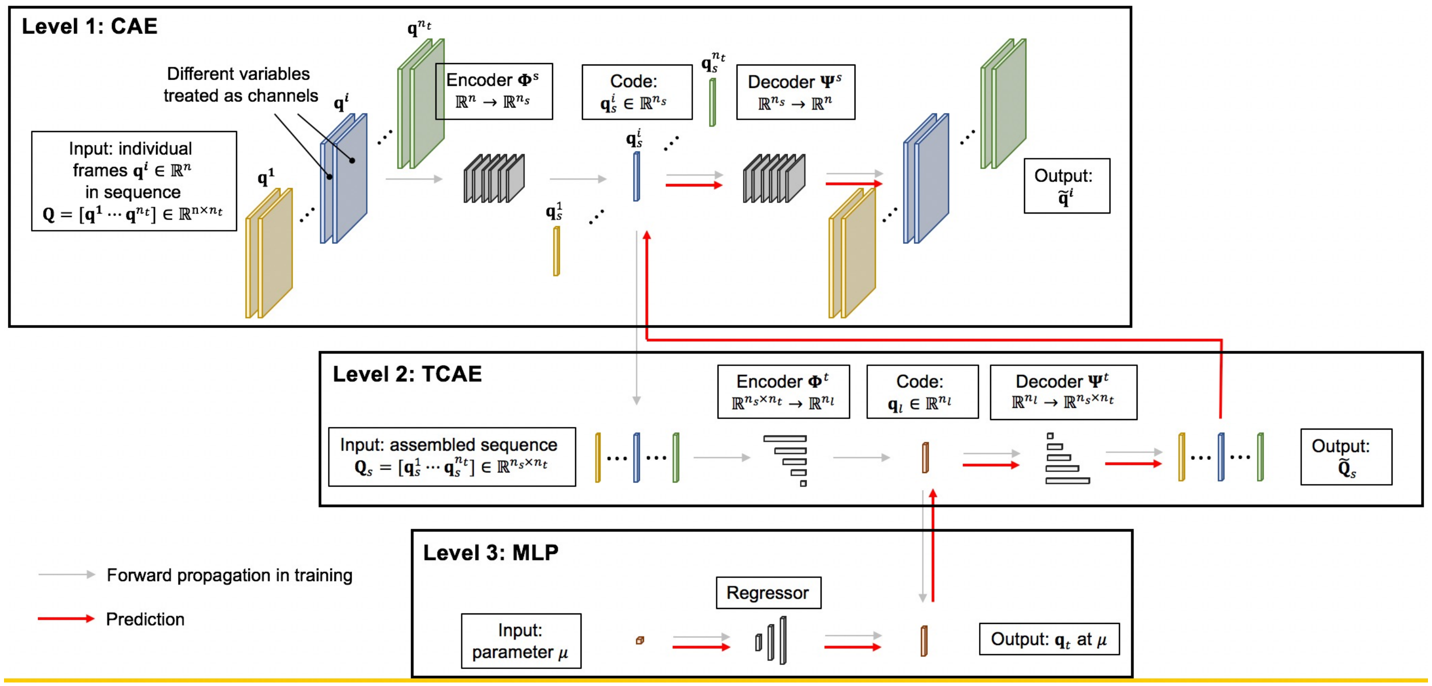
Acknowledgment

Multilevel Autoencoder networks



J Xu, K Duraisamy [Multi-level convolutional autoencoder networks for parametric prediction of spatio-temporal dynamics] Computer Methods in Applied Mechanics and Engineering 372, 2019

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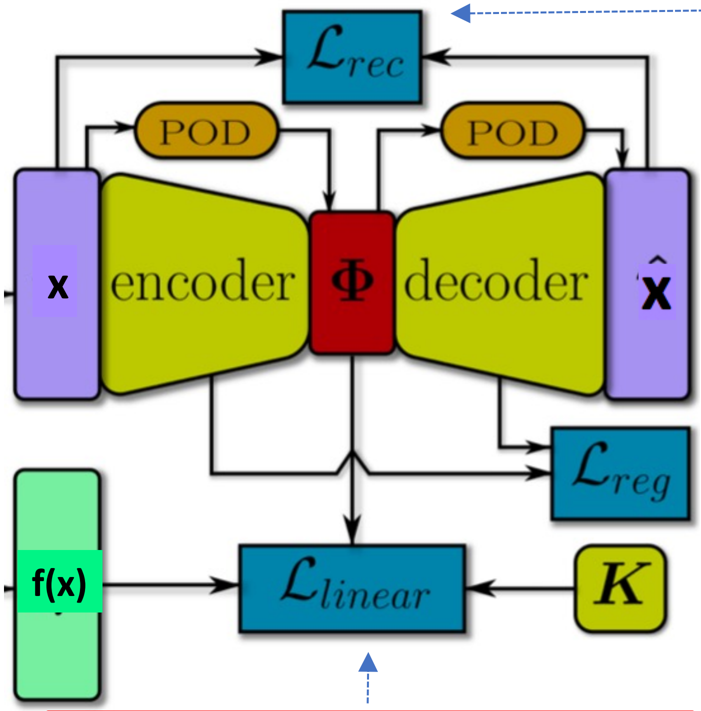
Math + physics + structure for Learning



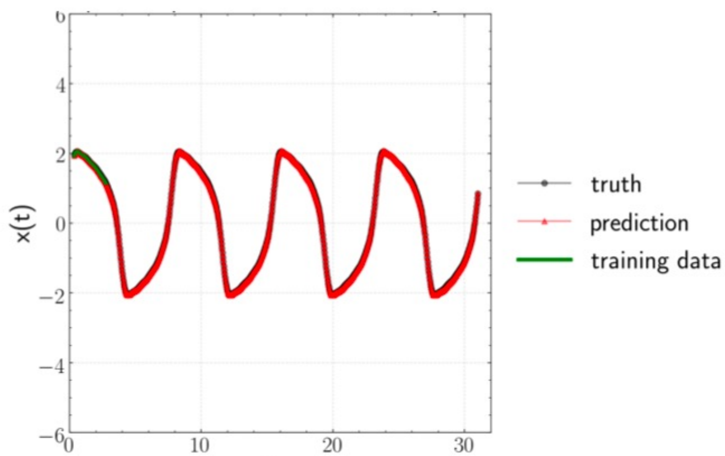
$$\frac{1}{M}\sum_{i=1}^{M}\|\mathbf{\Psi}(\mathbf{\Phi}(\mathbf{x}_i)) - \mathbf{x}_i\|^2$$

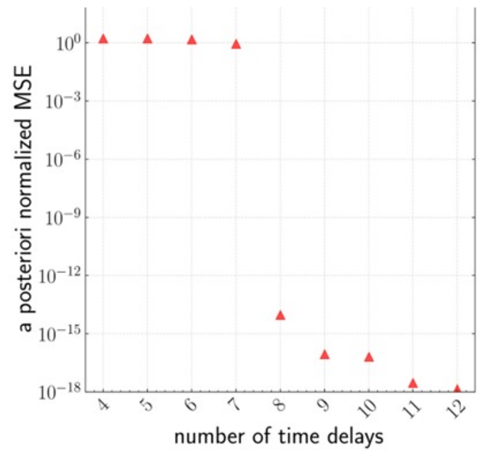
$$\mathbf{\Phi}(\boldsymbol{x}) = \mathbf{\Phi}_{dmd}(\boldsymbol{x}) + \mathbf{\Phi}_{nn}(\boldsymbol{x}),$$
$$\mathbf{\Psi}(\mathbf{\Phi}) = \mathbf{\Psi}_{dmd}(\mathbf{\Phi}(\boldsymbol{x})) + \mathbf{\Psi}_{nn}(\mathbf{\Phi}(\boldsymbol{x}))$$

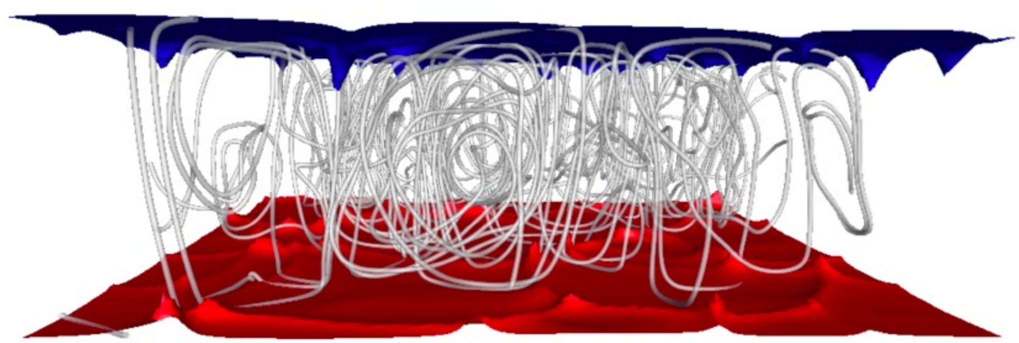
$$\frac{1}{M}\sum_{i=1}^{M}\|\mathbf{f}(\mathbf{x}_i)\cdot\nabla_{\mathbf{x}}\mathbf{\Phi}(\mathbf{x}_i) - \mathbf{\Phi}(\mathbf{x}_i)\mathbf{K}\|^2$$

Pan. S. & Duraisamy, K., *Physics-Informed Probabilistic Learning of Linear Embeddings of Non-linear Dynamics With Guaranteed Stability*, SIAM J. of Applied Dynamical Systems, 2020.

Power of linear embedding $\hat{\mathbf{x}}_{j+1} = \mathbf{W}_0\mathbf{x}_j + \mathbf{W}_1\mathbf{x}_{j-1} + \ldots + \mathbf{W}_L\mathbf{x}_{j-L},$







On the Structure of Time-delay Embedding in Linear Models of Non-linear Dynamical Systems
Pan & Duraisamy, Chaos 2020

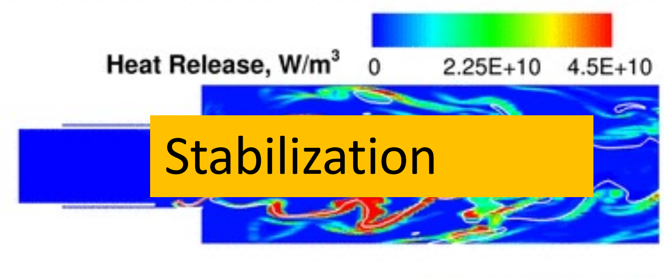
Metrics for Reduced order models

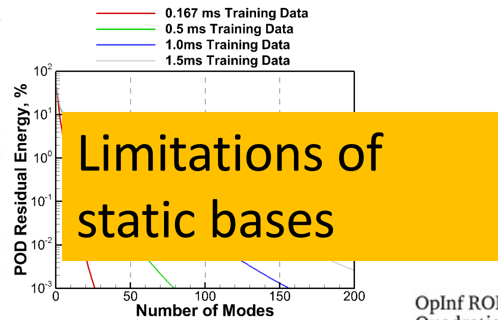
- Accuracy
- Robustness
- Realizability
- `True’ Predictivity*
- Efficiency*
- Time & complexity of development*
- Portability*
- Data requirements
  - ➔ Type of data
  - ➔ Amount of data

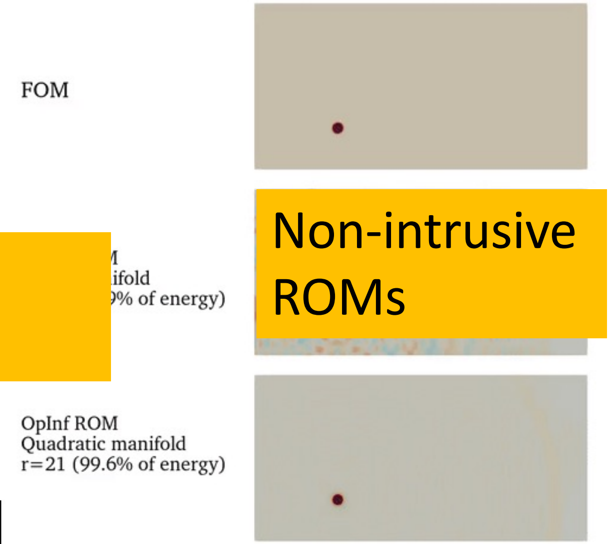
* Non-intrusive ROMs clearly win here

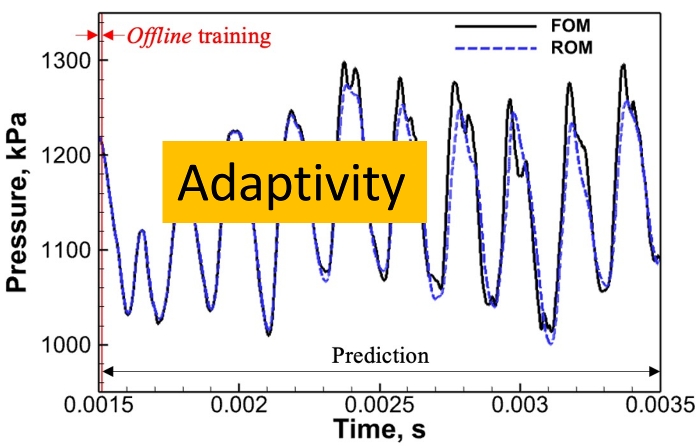
* Adaptive intrusive ROMs

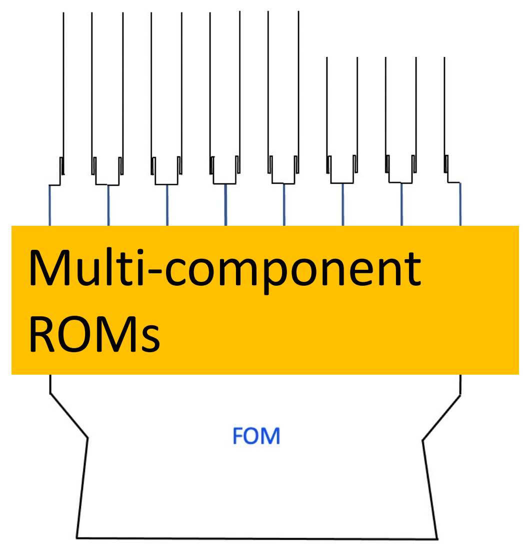
A few things to take home











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Projection-based ROMs

Full order model

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{dt}} = \mathbf{f}(\mathbf{q}, \mathrm{t}), \quad \mathbf{q}(0) = \mathbf{q}_0,$$

define a trial basis $\mathbf{V} \in \mathbb{R}^{n \times k}$ that spans a subspace $\mathcal{V} \subset \mathbb{R}^n$.

$$\mathbf{q} = \mathbf{V}\mathbf{q}_r + \mathbf{V}^{\perp}\mathbf{q}_p.$$

Approximate

$$\tilde{\mathbf{q}} = \mathbf{V}\mathbf{q}_r$$

Let's define a test basis $\mathbf{W}$ and project the equation onto the test subspace,

Project

$$\mathbf{W}^T\mathbf{V}\frac{d\mathbf{q}_r(t)}{dt} = \mathbf{W}^T\mathbf{f}(\mathbf{V}\mathbf{q}_r(t), t), \quad \mathbf{W}^T\mathbf{V}\mathbf{q}_r(0) = \mathbf{W}^T\mathbf{q}_0$$

ROM

$$\frac{d\mathbf{q}_r(t)}{dt} = [\mathbf{W}^T\mathbf{V}]^{-1}\mathbf{W}^T\mathbf{f}(\mathbf{V}\mathbf{q}_r(t), t), \quad \mathbf{q}_r(0) = [\mathbf{W}^T\mathbf{V}]^{-1}\mathbf{W}^T\mathbf{q}_0.$$

Equivalent FOM

$$\frac{d\tilde{\mathbf{q}}(t)}{dt} = \mathbf{V}[\mathbf{W}^T\mathbf{V}]^{-1}\mathbf{W}^T\mathbf{f}(\tilde{\mathbf{q}}(t), t), \quad \tilde{\mathbf{q}}(0) = \mathbf{V}[\mathbf{W}^T\mathbf{V}]^{-1}\mathbf{W}^T\mathbf{q}_0.$$

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Galerkin ROMs

Full Order Model

$$\frac{d\mathbf{q}}{dt} = \mathbf{f}(\mathbf{q},t), \quad \mathbf{q}(0) = \mathbf{q}_0, \qquad \mathbf{q} : [0,T] \rightarrow \mathbb{R}^N$$

define a trial basis $\mathbf{V} \in \mathbb{R}^{n\times k}$ that spans a subspace $\mathcal{V} \subset \mathbb{R}^n$.

$$\mathbf{q} = \mathbf{V}\mathbf{q}_r + \mathbf{V}^{\perp}\mathbf{q}_p.$$

Approximate

$$\tilde{\mathbf{q}} = \mathbf{V}\mathbf{q}_r$$

Substitute

$$\frac{d\mathbf{V}\mathbf{q}_r(t)}{dt} = \mathbf{f}(\mathbf{V}\mathbf{q}_r(t),t), \quad \mathbf{V}\mathbf{q}_r(0) = \mathbf{q}_0$$

Project

$$\frac{d\mathbf{q}_r(t)}{dt} = \mathbf{V}^T\mathbf{f}(\mathbf{V}\mathbf{q}_r(t),t), \quad \mathbf{q}_r(0) = \mathbf{V}^T\mathbf{q}_0.$$

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Error Transport

$$\begin{aligned}\boldsymbol{\epsilon}(t) &= \mathbf{q}(t) - \tilde{\mathbf{q}}(t) \\ &= \mathbf{q}(t) - \mathbf{V}\mathbf{q}_r(t) \\ &= \mathbf{q}(t) - \boldsymbol{\Pi}\mathbf{q}(t) + \boldsymbol{\Pi}\mathbf{q}(t) - \mathbf{V}\mathbf{q}_r(t) \\ &= [(\mathbf{I} - \boldsymbol{\Pi})\mathbf{q}(t)] + [\boldsymbol{\Pi}\mathbf{q}(t) - \mathbf{V}\mathbf{q}_r(t)] \\ &= \boldsymbol{\epsilon}_{\boldsymbol{\Pi}}(t) + \boldsymbol{\epsilon}_{\parallel}(t).\end{aligned}$$

$$\frac{d\boldsymbol{\epsilon}_{\parallel}}{dt} = \boldsymbol{\Pi}\left[\mathbf{f}(\mathbf{q}(t), t) - \mathbf{f}(\tilde{\mathbf{q}}(t), t)\right]$$

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Stability

Let's consider an autonomous linear system $\mathbf{f}(\mathbf{q}(t), t) = \mathbf{A}\mathbf{q}(t)$, then

$$\frac{d\boldsymbol{\epsilon}_{\parallel}}{dt} = \mathbf{\Pi A}\boldsymbol{\epsilon}_{\parallel}(t) + \mathbf{\Pi A}\boldsymbol{\epsilon}_{\mathbf{\Pi}}(t).$$

$$\begin{aligned}
\boldsymbol{\epsilon}_{\parallel}^{T}\frac{d\boldsymbol{\epsilon}_{\parallel}}{dt} &= \boldsymbol{\epsilon}_{\parallel}^{T}\mathbf{\Pi A}\boldsymbol{\epsilon}_{\parallel} + \boldsymbol{\epsilon}_{\parallel}^{T}\mathbf{\Pi A}\boldsymbol{\epsilon}_{\mathbf{\Pi}} \\
\frac{1}{2}\frac{d\boldsymbol{\epsilon}_{\parallel}^{T}\boldsymbol{\epsilon}_{\parallel}}{dt} &= \frac{1}{2}\boldsymbol{\epsilon}_{\parallel}^{T}[\mathbf{\Pi A} + [\mathbf{\Pi A}]^{T}]\boldsymbol{\epsilon}_{\parallel} + \boldsymbol{\epsilon}_{\parallel}^{T}\mathbf{\Pi A}\boldsymbol{\epsilon}_{\perp}
\end{aligned}$$

we get the necessary condition [2], that $\mathbf{\Pi A} + [\mathbf{\Pi A}]^{T}$ should be negative definite. Additionally, the interaction between the parallel and orthogonal errors may also affect stability in a profound manner.

In Galerkin ROMs, we do not have a great degree of control over $\mathbf{\Pi}$. Petrov Galerkin methods give us additional control knobs to improve both accuracy and stability.

Galerkin & LSPG

FOM

$$\mathbf{q}^n = \mathbf{q}^{n-1} + \Delta t \mathbf{A}\mathbf{q}^n$$

Galerkin ROM

$$\left[\mathbf{I} - \Delta t \mathbf{V}^T \mathbf{A}\mathbf{V} \qquad\qquad\qquad \right] \mathbf{q}_r^n = \left[\mathbf{I} \qquad\qquad \right] \mathbf{q}_r^{n-1}$$

Least squares projection

$$\min_{\mathbf{q}_r^n} ||\mathbf{V}\mathbf{q}_r^n - \mathbf{V}\mathbf{q_r}^{n-1} - \Delta t \mathbf{A}\mathbf{V}\mathbf{q}^n||_2^2$$

Farhat and co-workers, circa 2010

LSPG ROM

$$\left[\mathbf{I} - \Delta t \mathbf{V}^T \mathbf{A}\mathbf{V} - \Delta t \mathbf{V}^T \mathbf{A}^T \mathbf{V} + \Delta t^2 \mathbf{V}^T \mathbf{A}^T \mathbf{A}\mathbf{V}\right] \mathbf{q}_r^n = \left[\mathbf{I} - \Delta t \mathbf{V}^T \mathbf{A}^T \mathbf{V}\right] \mathbf{q}_r^{n-1}$$

Linear Manifold ROMs

Continuous FOM

$$\frac{d\mathbf{q}}{dt} = \mathbf{f}(\mathbf{q}, t), \quad \mathbf{q}(0) = \mathbf{q}_0, \qquad \mathbf{q} : [0, T] \rightarrow \mathbb{R}^N$$

Discrete FOM

$$\mathbf{r}(\mathbf{q}^n) \triangleq \mathbf{q}^n + \sum_{j=1}^{l} \alpha_j \mathbf{q}^{n-j} - \Delta t \beta_0 \mathbf{f}(\mathbf{q}^n, t^n) - \Delta t \sum_{j=1}^{l} \beta_j \mathbf{f}(\mathbf{q}^{n-j}, t^{n-j})$$

Reduced representation in Linear subspace

$$\tilde{\mathbf{q}} = \mathbf{q}_{\text{ref}} + \mathbf{P}^{-1} \mathbf{V} \mathbf{q}_r$$

$\tilde{\mathbf{q}}$: $N \times 1$ $\mathbf{P}^{-1}$: $N \times N$ $\mathbf{V}$: $N \times k$ $\mathbf{q}_r$: $k \times 1$

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Linear Stability : 1/2

Continuous FOM

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{dt}} = \mathbf{J}\mathbf{q}, \quad \mathbf{q}(0) = \mathbf{q}_0,$$

Discrete FOM

$$(\mathbf{I} - \Delta t \mathbf{J})\mathbf{q}^n = \mathbf{q}^{n-1}, \quad \mathbf{q}^0 = \mathbf{q}_0$$

Decomposition

$$\tilde{\mathbf{q}}(\mathrm{t}) \triangleq \mathbf{V}\mathbf{q}_r(\mathrm{t}), \text{ where } \mathbf{V} \in \mathbb{R}^{N \times k} \text{ and } \mathbf{q}_r \in \mathbb{R}^k$$

Theorem 1 : *If the Discrete FOM above is asymptotically stable in the sense of* $\left\|(\mathbf{I} - \Delta t \mathbf{J})^{-1}\right\|_2 \leq 1$ *, then the Backward Euler Galerkin ROM is also asymptotically stable if* $\lambda_n\left(\mathbf{I} - 0.5\Delta t(\mathbf{J} + \mathbf{J}^T)\right) \geq 1$

Model Reduction for Multi-Scale Transport Problems using Structure-Preserving Least-Squares Projections with Variable Transformation C Huang, C Wentland, K Duraisamy, C Merkle, JCP 2021.

Linear Stability : 2/2

Continuous FOM

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \mathbf{J}\mathbf{q}, \quad \mathbf{q}(0) = \mathbf{q}_0,$$

Discrete FOM

$$(\mathbf{I} - \Delta t \mathbf{J})\mathbf{q}^n = \mathbf{q}^{n-1}, \quad \mathbf{q}^0 = \mathbf{q}_0$$

Decomposition

$$\tilde{\mathbf{q}}(\mathrm{t}) \triangleq \mathbf{V}\mathbf{q}_r(\mathrm{t}), \text{ where } \mathbf{V} \in \mathbb{R}^{N \times k} \text{ and } \mathbf{q}_r \in \mathbb{R}^k$$

Theorem 2 : *If the Discrete FOM above is asymptotically stable in the sense of* $\left\|(\mathbf{I} - \Delta t \mathbf{J})^{-1}\right\|_2 \leq 1$ *, then the associated LSPG ROM is also asymptotically stable with no further assumptions required.*

Model Reduction for Multi-Scale Transport Problems using Structure-Preserving Least-Squares Projections with Variable Transformation C Huang, C Wentland, K Duraisamy, C Merkle, JCP 2021.

Linear Manifold Projection-based ROMs

$$\frac{d\mathbf{q}}{dt} = \mathbf{f}(\mathbf{q}, t), \quad \mathbf{q}(0) = \mathbf{q}_0, \qquad \mathbf{q} : [0, T] \to \mathbb{R}^N$$

$$\mathbf{r}(\mathbf{q}^n) \triangleq \mathbf{q}^n + \sum_{j=1}^{l} \alpha_j \mathbf{q}^{n-j} - \Delta t \beta_0 \mathbf{f}(\mathbf{q}^n, t^n) - \Delta t \sum_{j=1}^{l} \beta_j \mathbf{f}\left(\mathbf{q}^{n-j}, t^{n-j}\right)$$

$$\mathcal{V} \triangleq \text{Range}(\mathbf{P}^{-1}\mathbf{V})$$

Projection

$$\frac{d\mathbf{q}_r}{dt} = \mathbf{V}^T \mathbf{P} \mathbf{f}(\tilde{\mathbf{q}}, t)$$

Galerkin

$$\tilde{\mathbf{q}}^n \triangleq \arg \min_{\tilde{\mathbf{q}}^n \in \text{Range}(\mathbf{V})} \|\mathbf{P}\mathbf{r}(\tilde{\mathbf{q}}^n)\|_2^2$$

Least Squares

Multi-scale, Multi-physics, Complexity : An Example

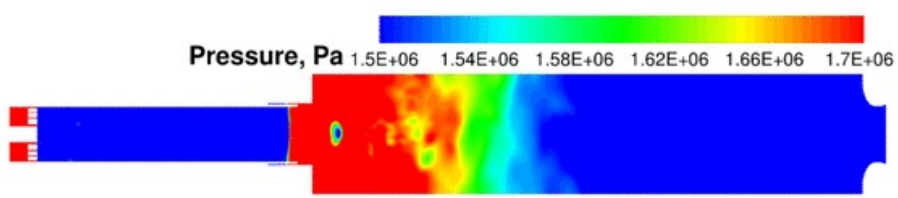
- Non-linear, Multi-scale multi-physics interactions : acoustics, flow & reaction
- Flow – Large coherent structures + small shear layer dynamics
- Reaction – Highly intensive, distributed & intermittent thin flame
- High sensitivity to parameter changes

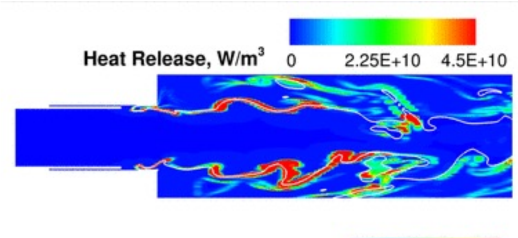
$$\frac{\partial Q}{\partial t} + \frac{\partial F_i}{\partial x_i} + \frac{\partial F_{v,i}}{\partial x_i} = H$$

$$Q = \begin{pmatrix} \rho \\ \rho u_i \\ \rho h^0 - p \\ \rho Y_l \end{pmatrix}, \; F_i = \begin{pmatrix} \rho u_i \\ \rho u_i u_j \\ \rho u_i h^0 \\ \rho u_i Y_l \end{pmatrix}, \; F_{v,i} = \begin{pmatrix} 0 \\ \tau_{ij} \\ u_j \tau_{ji} + q_i \\ \rho V_{i,l} Y_l \end{pmatrix}, \; H = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dot{\omega}_l \end{pmatrix}$$

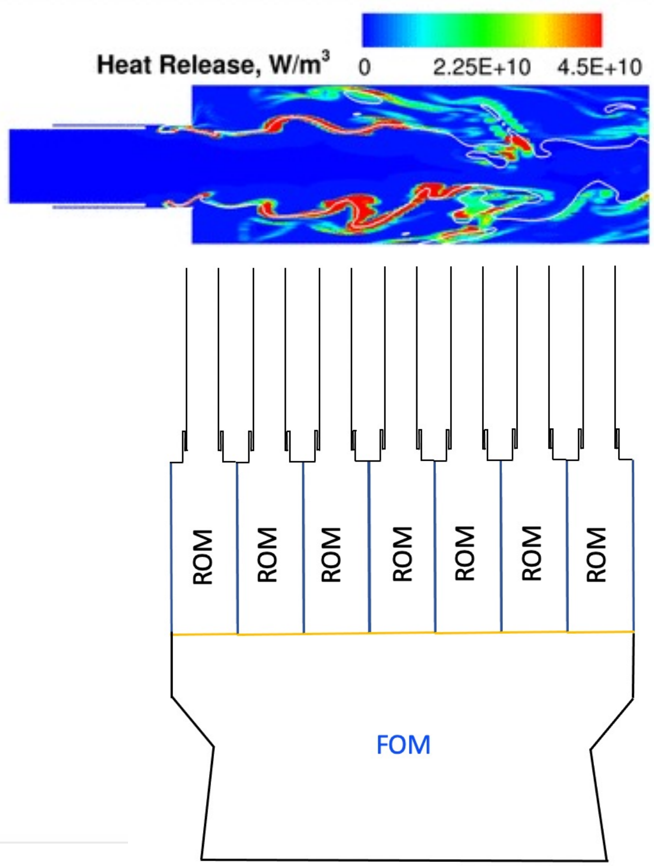
Highly nonlinear and stiff source term :

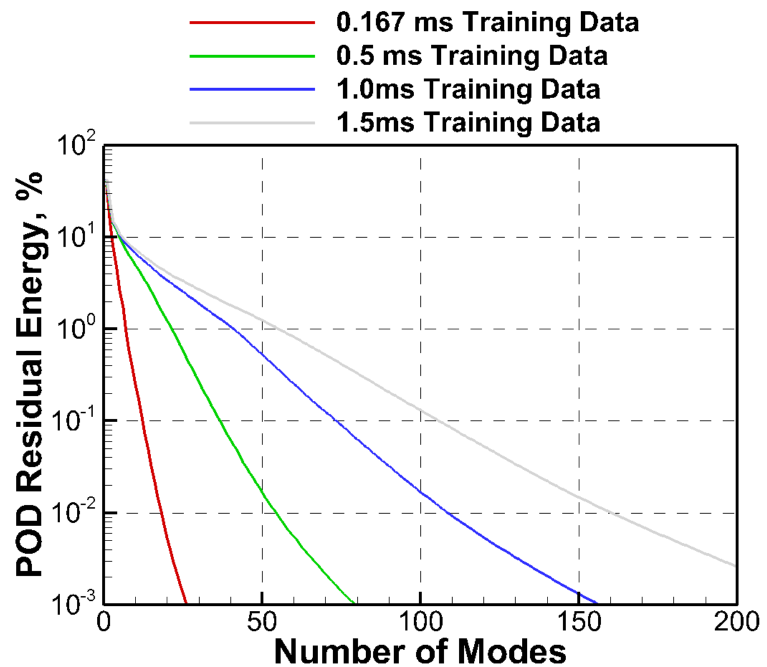
$$e.g., \; \dot{\omega}_1 = \frac{\rho Y_1}{M_1} A T^b \exp\left(\frac{-E_a}{R_u T}\right) [\frac{\rho Y_1}{M_1}]^{0.2} [\frac{\rho Y_2}{M_2}]^{1.3}$$





3 Things to take home





Key Enablyer : Adaptivity

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Part 1

Robustness & Accuracy

Model Reduction for Multi-Scale Transport Problems using Structure-Preserving Least-Squares Projections with Variable Transformation *C Huang, C Wentland, K Duraisamy, JCP, 2021*

Discretely-consistent Petrov-Galerkin ROMs with Variable Transformation (1/3)

Conservative Equations

$$\frac{\mathbf{q}^{n+1} - \mathbf{q}^{n}}{\Delta t} = \mathbf{f}(\mathbf{q}^{n+1})$$

Pseudo time stepping

Conserved variables

$$\delta \mathbf{q} = \Gamma \delta \mathbf{s}$$

Primitive variables

$$\Gamma^{p-1} \frac{\mathbf{s}^{p} - \mathbf{s}^{p-1}}{\Delta \tau} + \frac{\mathbf{q}^{p} - \mathbf{q}^{n}}{\Delta t} = \mathbf{f}(\mathbf{q}^{p})$$

Fully Discrete Residual

$$\mathbf{r}_{\mathbf{s}}(\mathbf{s}^{p}) \triangleq \left[\left(\frac{\Delta t}{\Delta \tau} + 1 \right) \mathbf{I} - \Delta t \mathbf{J}^{p-1} \right] \Gamma^{p-1} (\mathbf{s}^{p} - \mathbf{s}^{p-1}) + (\mathbf{q}^{p-1} - \mathbf{q}^{n}) - \Delta t \mathbf{f}(\mathbf{q}^{p-1})$$

Discretely-consistent Petrov-Galerkin ROMs with Variable Transformation (2/3)

Discrete equations

$$\mathbf{r}_{\mathbf{s}}(\mathbf{s}^p) \triangleq \left[\left(\frac{\Delta t}{\Delta \tau}+1\right)\mathbf{I}-\Delta t\mathbf{J}^{p-1}\right]\Gamma^{p-1}(\mathbf{s}^p-\mathbf{s}^{p-1})+(\mathbf{q}^{p-1}-\mathbf{q}^n)-\Delta t\mathbf{f}(\mathbf{q}^{p-1})$$

Decomposition

$$\tilde{\mathbf{s}} \triangleq \bar{\mathbf{s}}+\mathbf{S}\mathbf{V}\mathbf{s}_r$$

- $\mathbf{S}$: Scaling matrix for $\mathbf{s}$
- $\mathbf{V}$: Basis vectors
- $\mathbf{s}_r$: Basis coefficients

Discretely-consistent Least-Squares formulation

$$\mathbf{s}_r{}^p \triangleq \arg\min_{\mathbf{s}_r} ||\mathbf{Q}\mathbf{r}_{\mathbf{s}}(\tilde{\mathbf{s}})||_2^2$$

- $\mathbf{Q}$: Scaling matrix for $\mathbf{q}$

Discretely-consistent Petrov-Galerkin ROMs with Variable Transformation (3/3)

Discretely-consistent Least-Squares formulation

$$\mathbf{s}_r{}^p \triangleq \arg\min_{\mathbf{s}_r} \|\mathbf{Q}\mathbf{r}_\mathbf{s}(\tilde{\mathbf{s}})\|_2^2$$

Test basis

$$\mathbf{W}^{p-1} \triangleq \mathbf{Q}\left[\left(\frac{\Delta t}{\Delta \tau}+1\right)\mathbf{I}-\Delta t\mathbf{J}^{p-1}\right]\Gamma^{p-1}\mathbf{S}\mathbf{V}$$

ROM

$$[\mathbf{W}^{p-1}]^T\ \mathbf{W}^{p-1}\ (\mathbf{s}_r{}^p-\mathbf{s}_r{}^{p-1}) = -[\mathbf{W}^{p-1}]^T\mathbf{Q}\mathbf{r}_\mathbf{q}{}^{p-1}$$

Discretely-consistent, symmetrized, (globally)stable, conservative

Hyper-reduction

Re-define discrete least square formulation based on N_s samples

$$[\mathbf{W}^{p-1}]^T \mathbf{W}^{p-1}(\mathbf{s}_r{}^p - \mathbf{s}_r{}^{p-1}) = -[\mathbf{W}^{p-1}]^T [\mathbf{PU}]^+ \mathbf{PQ}\mathbf{r}_\mathbf{q}{}^{p-1}$$

Sampling points (QDEIM)

Residual Basis (POD)

$$[\mathbf{PU}]^+\mathbf{PQ}\left[\left(\frac{\Delta t}{\Delta\tau}+1\right)\mathbf{I} - \Delta t \mathbf{J}^{p-1}\right]\Gamma^{p-1}\mathbf{SV}$$

N_s x k

k x N_s
precomputed

$$\mathbf{PQ}\left((\tilde{\mathbf{q}}^{p-1}-\tilde{\mathbf{q}}^n) - \Delta t\, \mathbf{f}(\tilde{\mathbf{q}}^{p-1})\right)$$

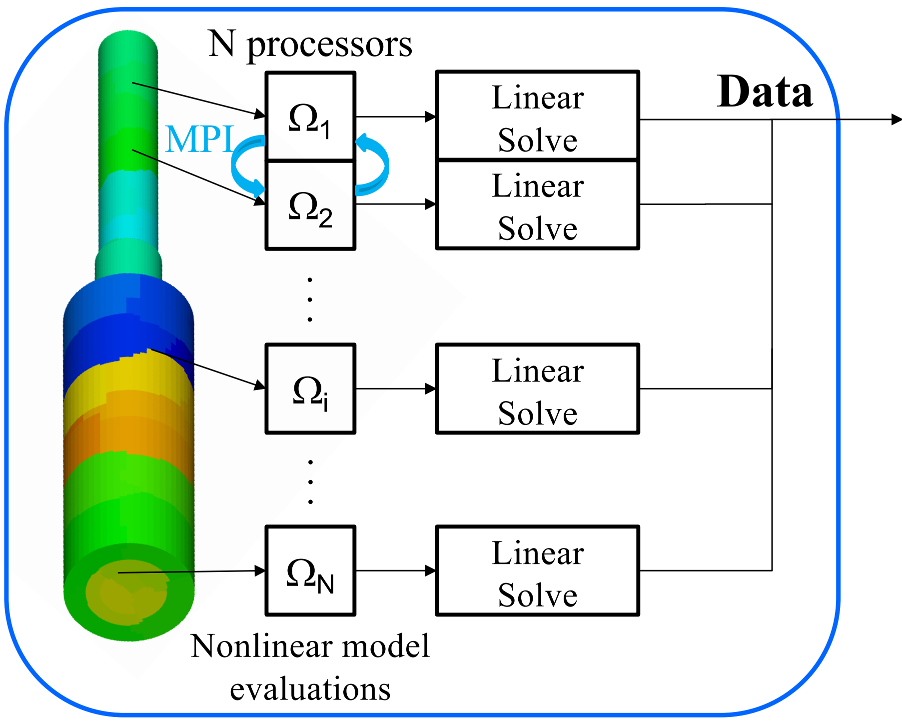
N_s x 1

Preserves discrete consistency, but reduces complexity from O(N) to O(N_s)

Parallel Data Processor (PDP) for Large scale ROMs

ROM Software Infrastructure Development → Efficient Data Processor

CFD Code



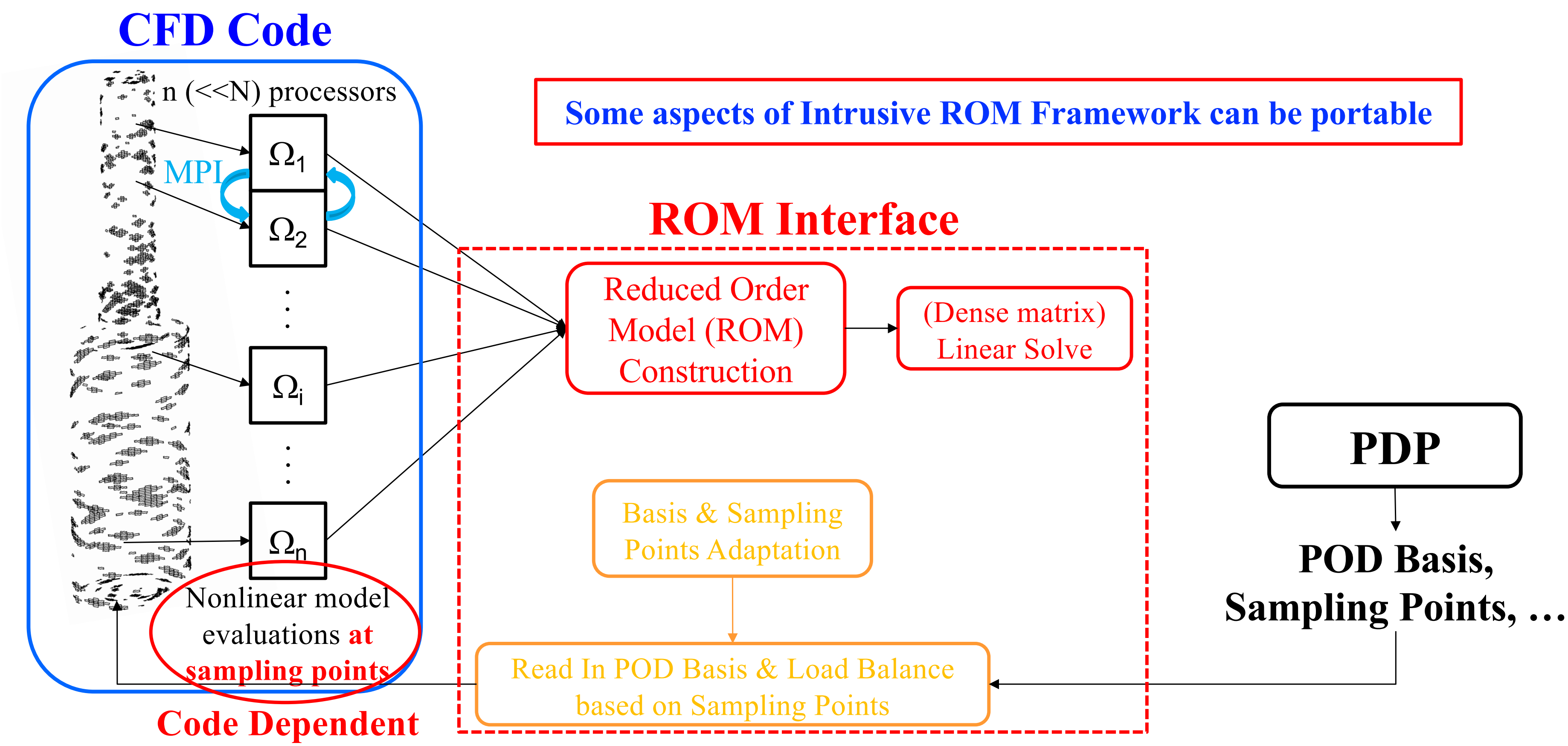
Parallel Data Processor (PDP)

- Parallel toolset for data decomposition (POD/DMD) – shared with Air Force
- Compatible with different data formats
- Efficient basis generation, sparse sampling pre-calculations especially for large scale problems (e.g. POD with 100M DoFs x 1000 snapshots in less than 30 min)

POD Basis, Sampling Points, …

ROM Interface

ROM Software Infrastructure Development → Portable Parallel Interface



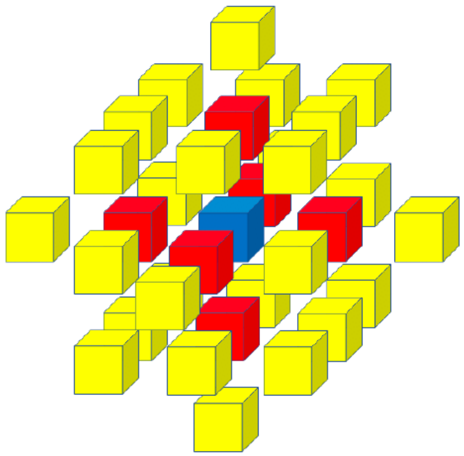
Further details on sampling

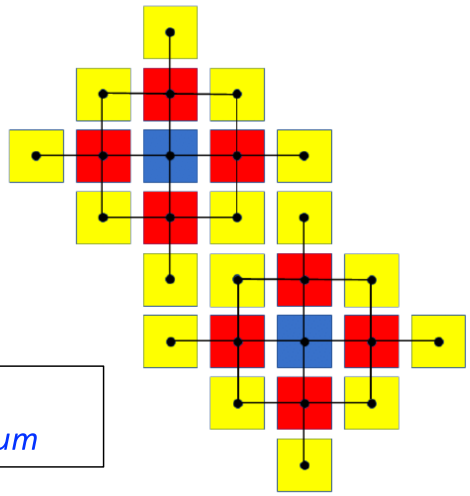
- First d points selected by QR pivots of data (QDEIM)
- **GappyPOD+R**: randomized oversampling
  - Remaining $N_s - d$ points are selected randomly
  - Cheap, simple, serial
- **GappyPOD+E**: eigenvector-based oversampling
  - Minimize sampling error at every iteration

$$\left|\left|[\mathbf{S}_m^T\mathbf{U}]^+\right|\right|_2 = \frac{1}{\sigma_{min}\left(\mathbf{S}_m^T\mathbf{U}\right)}$$

  - Sample row of **U** which maximizes update to smallest eigenvalue

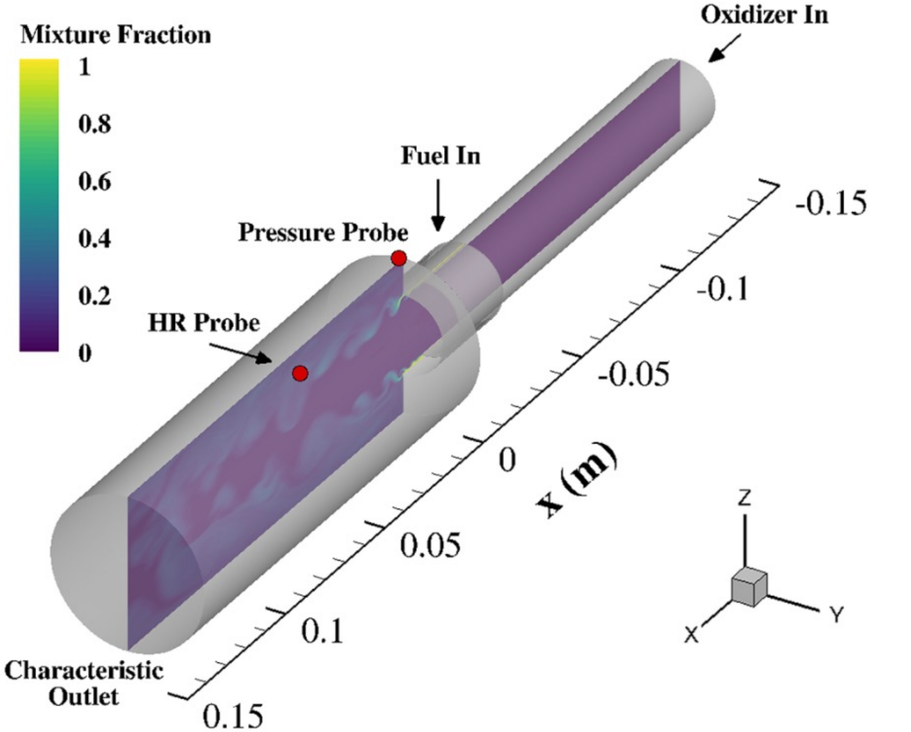
$$\lambda_d^{m+1} - \lambda_d^m$$





Investigation of Sampling Strategies for Reduced-Order Models of Rocket Combustors, CR Wentland, C Huang, K Duraisamy, Proc. AIAA Scitech 2021 Forum

Further details on sa



Scalable Projection-Based Reduced-Order Models for Large Multiscale Fluid Systems
, CR Wentland, C Huang, K Duraisamy, AIAAJ 2023

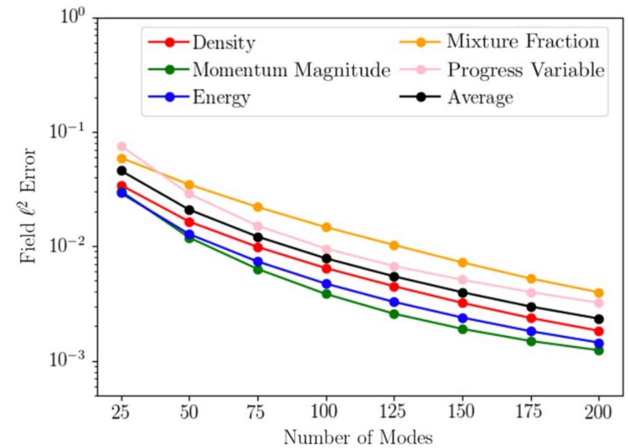


Fig. 19 Conservative variables time-average projection error.

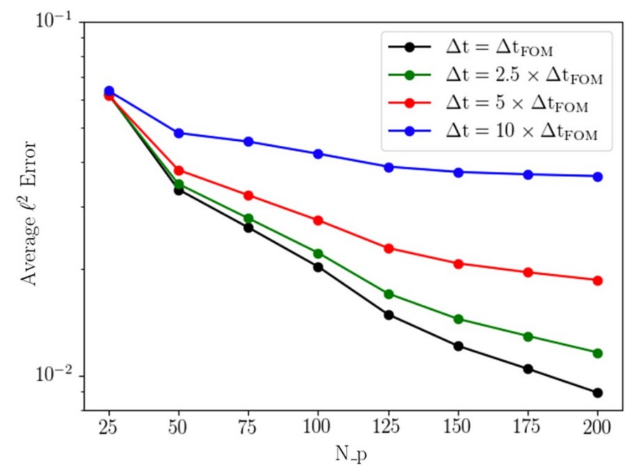
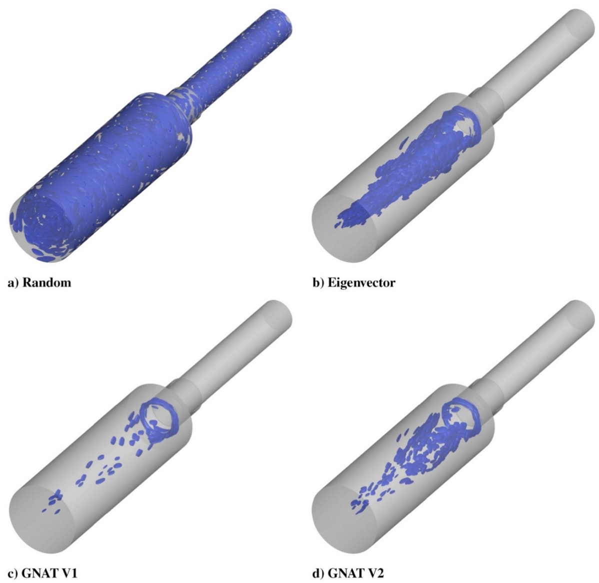
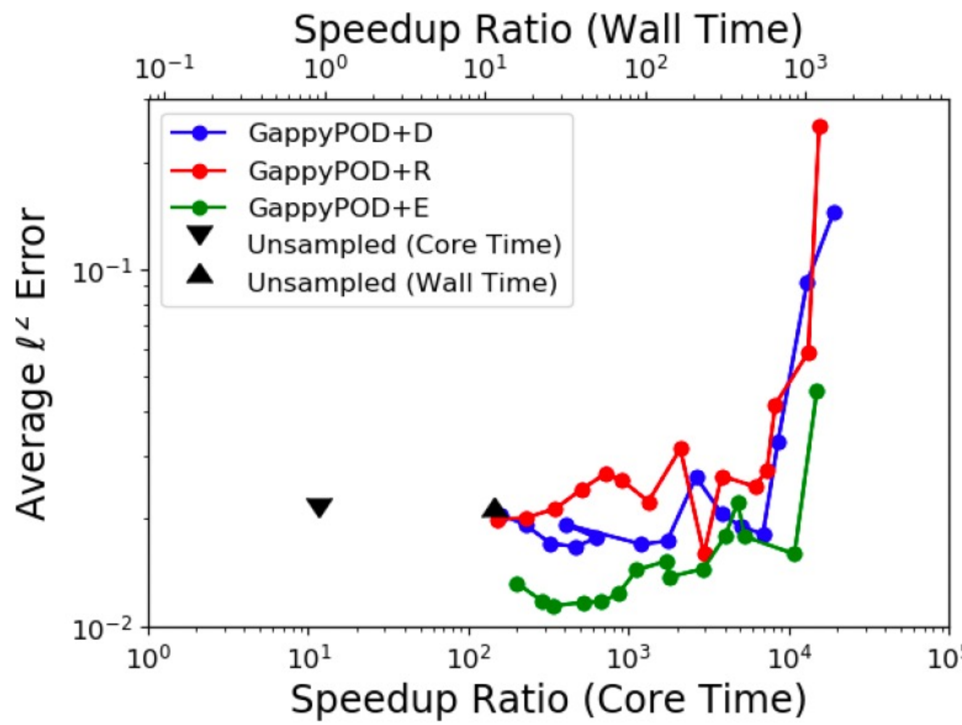


Fig. 20 CVRC unsampled PROM time-average error, various Δt.

Further details on sampling





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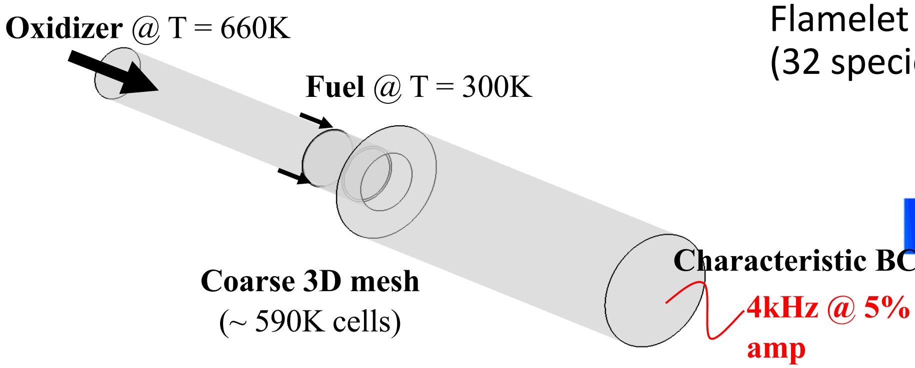
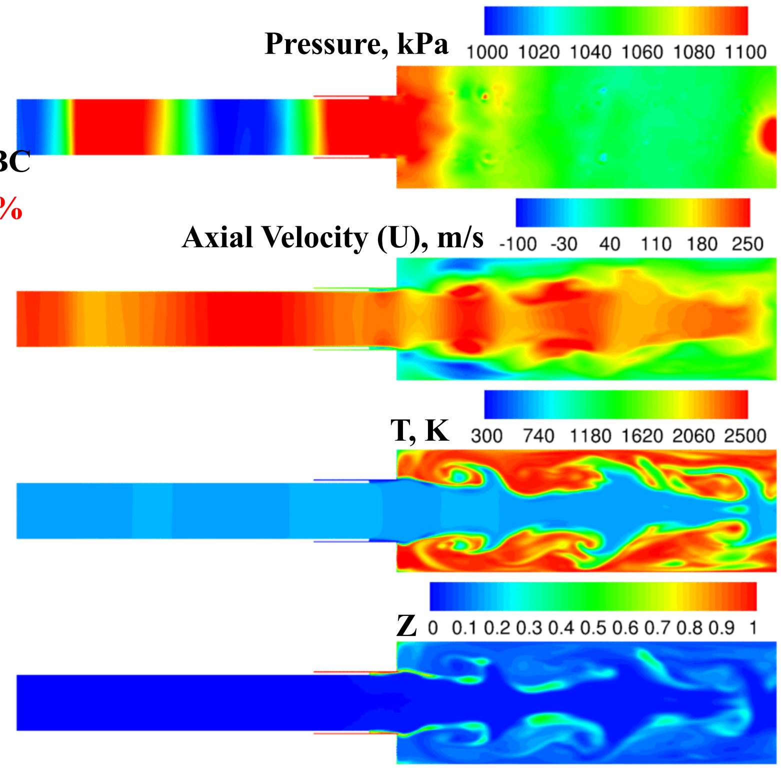
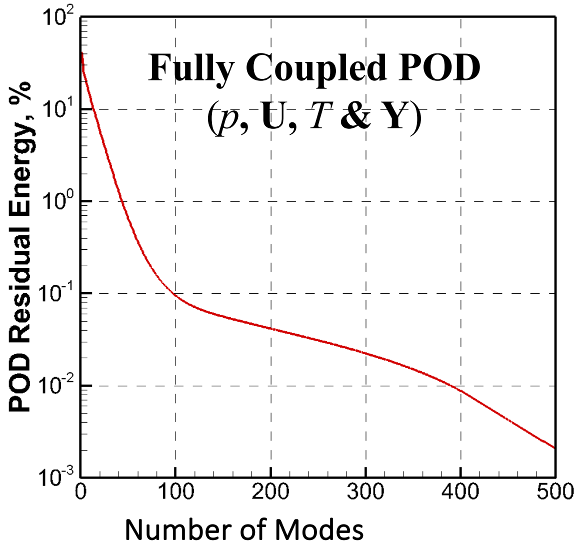
3D Truncated CVRC Injector with Downstream Forcing



Flamelet progress variable model with GRI-1.2 kinetics (32 species, 177 reactions)





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Introduction

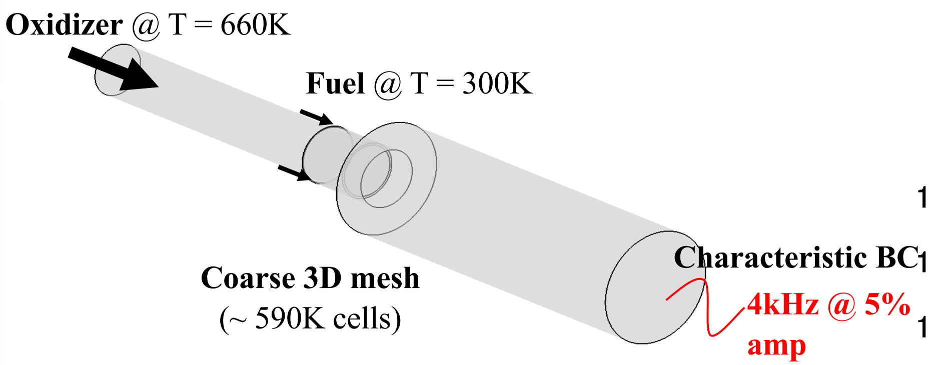
SP-LSVT ROMs
- Formulation
- Results

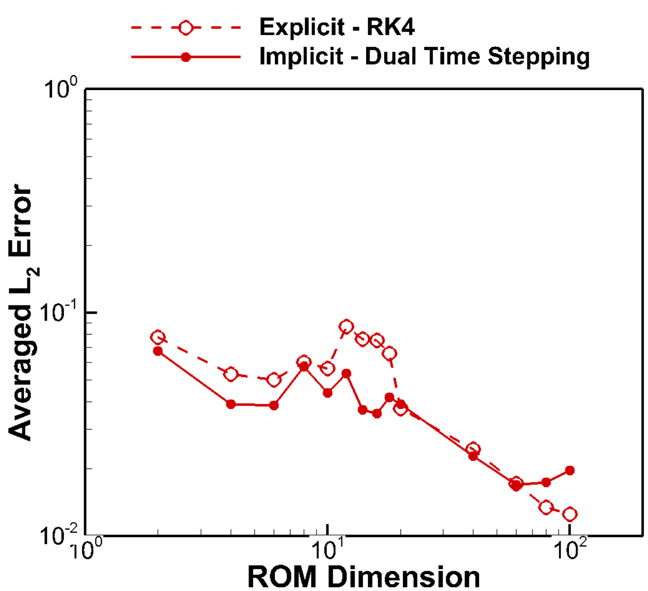
Adaptive Basis
- Formulation
- Results

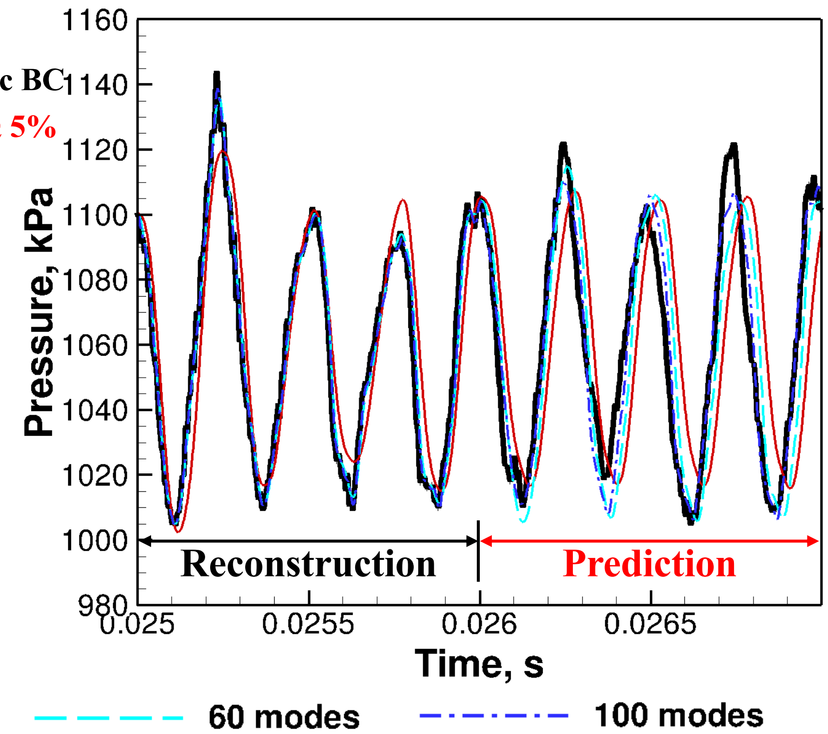
ROM Networks
- Formulation
- Results

Summary

Model Performance







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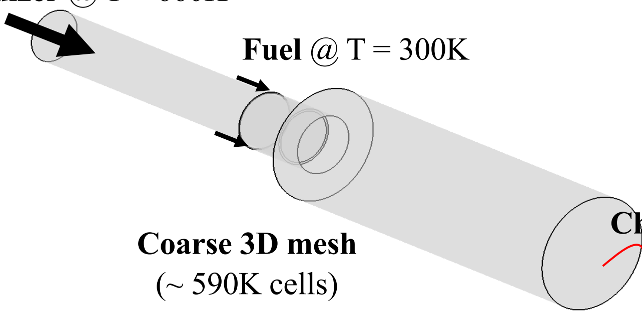
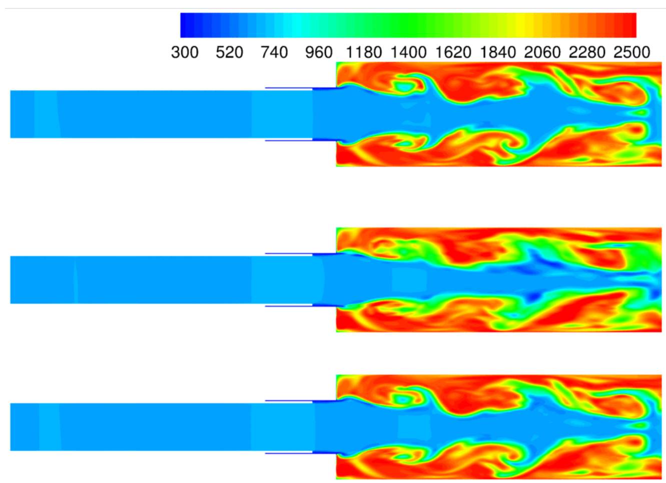


T, K

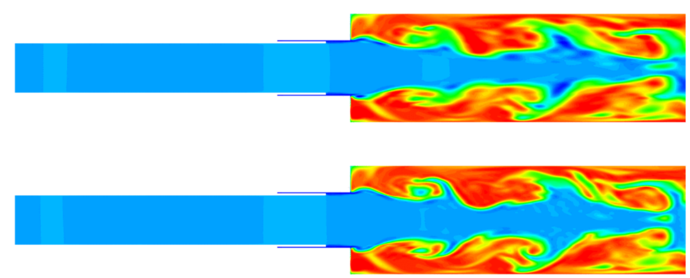
FOM

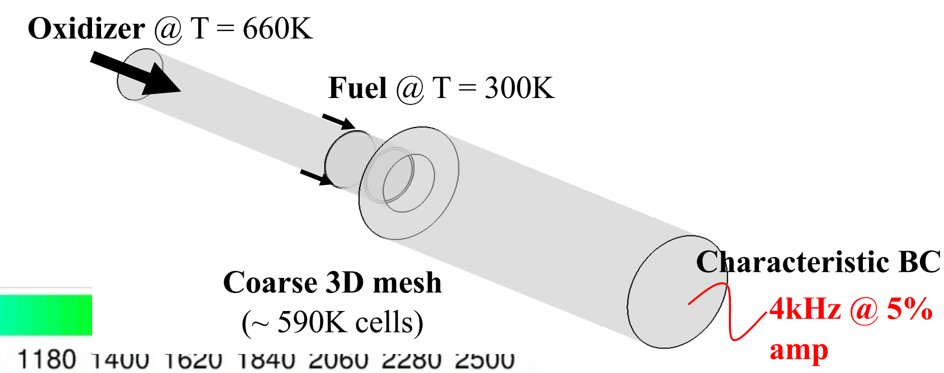
ROM 20 modes

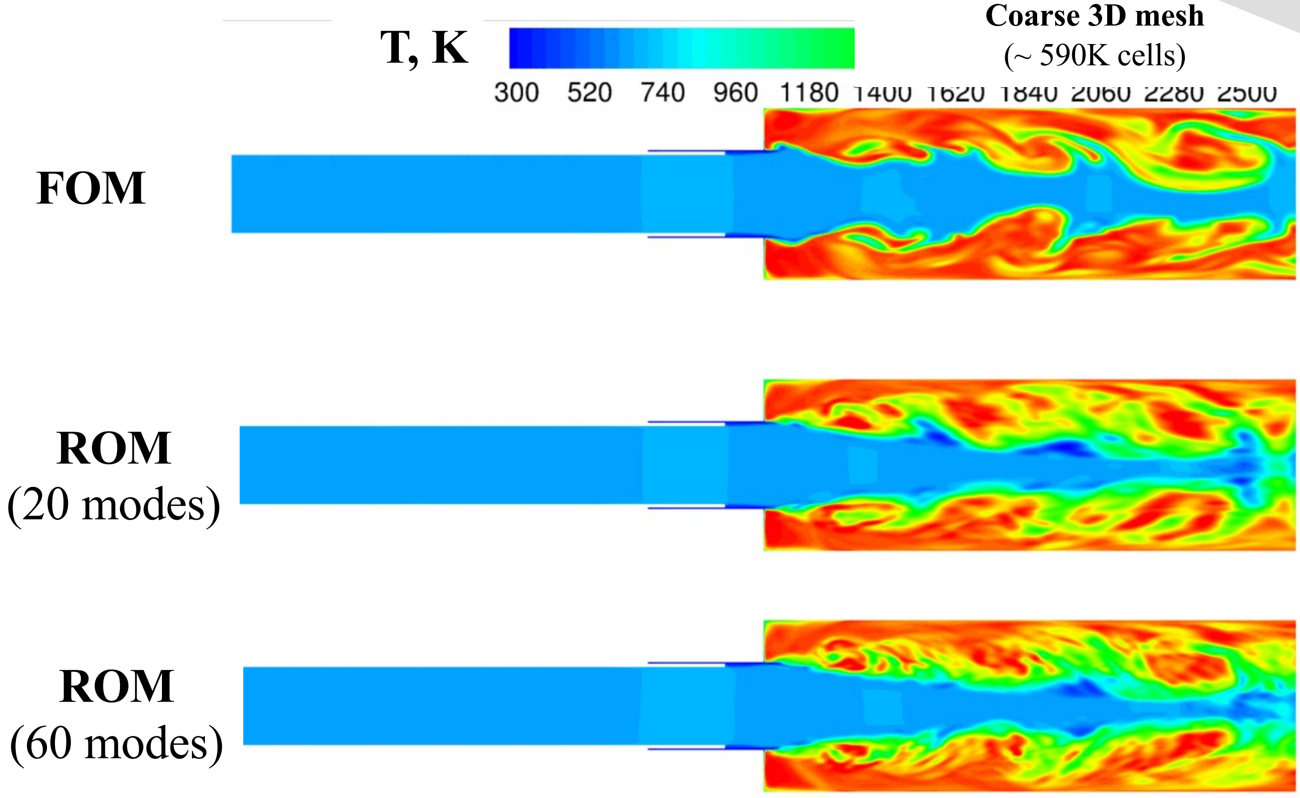
ROM 60 modes



Projected FOM

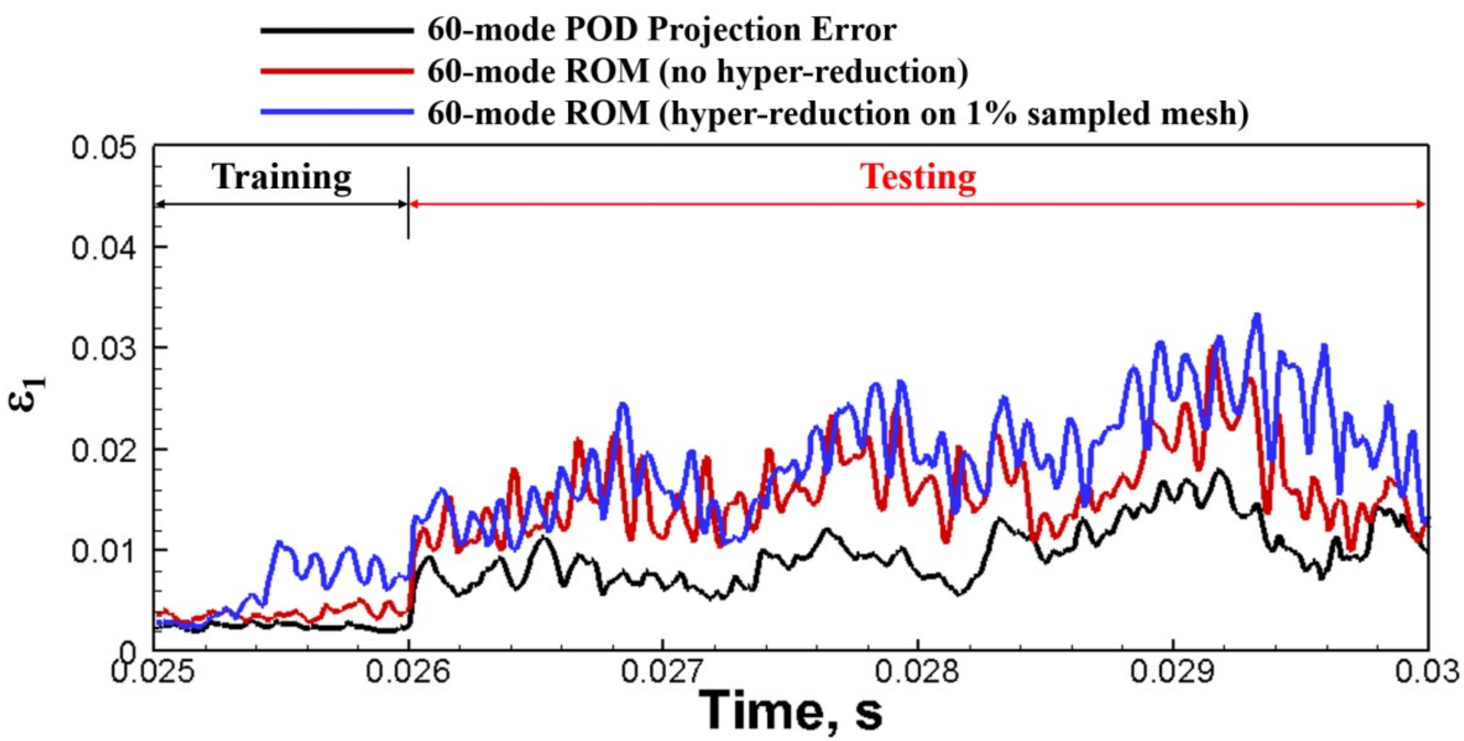




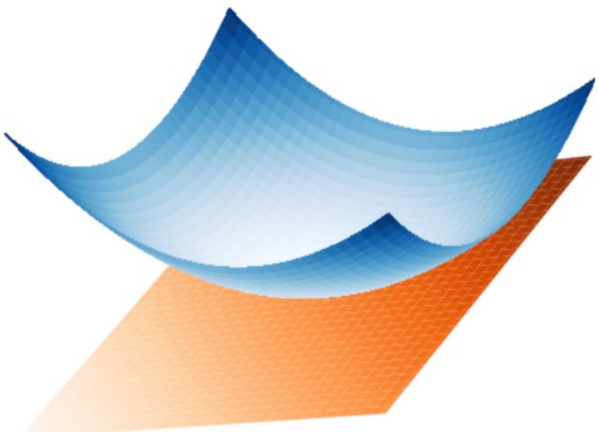


3-4 Orders of magnitude cost reduction

Impact of projection error and sampling



$$d_n(\mathcal{A}, \mathcal{X}) = \inf_{\mathcal{X}_n} \sup_{x \in \mathcal{A}} \inf_{y \in \mathcal{X}_n} ||x - y||_{\mathcal{X}}$$



Non-linear Manifold SP-LSVT

- Dual-time formulation w/r/t primitive state

$$\mathbf{\Gamma}(\mathbf{s})\frac{\partial \mathbf{s}}{\partial \tau} + \frac{\partial \mathbf{q}(\mathbf{s})}{\partial t} - \mathbf{f}(\mathbf{q}) = \mathbf{0}, \quad \mathbf{\Gamma} = \frac{\partial \mathbf{q}}{\partial \mathbf{s}}$$

- Introduce similar affine representation of primitive state

$$\mathbf{s} \approx \widetilde{\mathbf{s}} = \bar{\mathbf{s}} + \mathbf{S}\mathbf{\Psi}(\mathbf{s}_r) \quad ; \quad \mathbf{\Psi} : \mathbb{R}^K \rightarrow \mathbb{R}^N$$

-

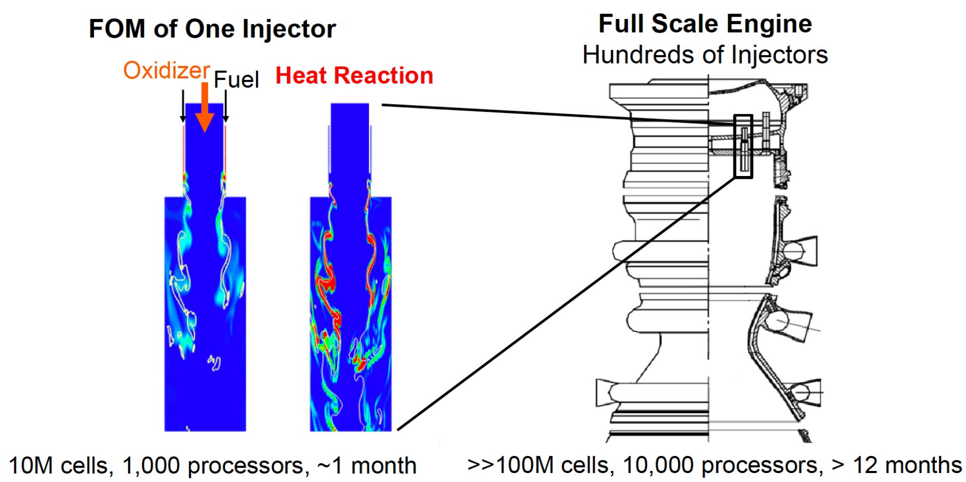
$$\mathbf{s}_r^n = \underset{\mathbf{a} \in \mathbb{R}^K}{\text{argmin}} \, ||\mathbf{Q}\mathbf{r}_s(\bar{\mathbf{s}} + \mathbf{S}\mathbf{\Psi}(\mathbf{a}))||_2^2$$

$$(\mathbf{W}^{p-1})^T \mathbf{W}^{p-1} \left[\mathbf{s}_r^p - \mathbf{s}_r^{p-1}\right] = -(\mathbf{W}^{p-1})^T \mathbf{r}_s(\widetilde{\mathbf{s}}^{p-1})$$

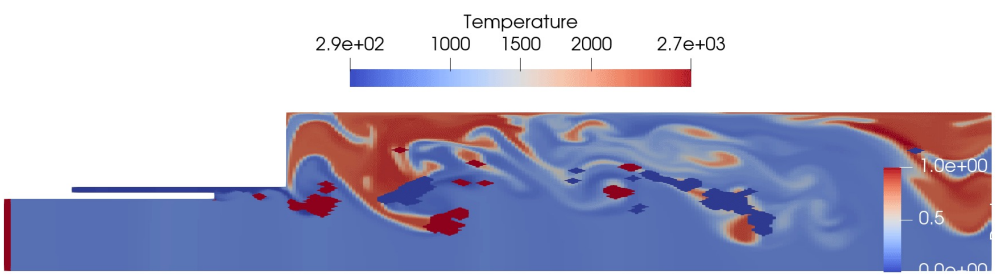
$$\mathbf{W}^{p-1} = \mathbf{Q}\frac{\partial \mathbf{r}_s(\widetilde{\mathbf{s}}^{p-1})}{\partial \widetilde{\mathbf{s}}} \mathbf{S} \left[\frac{\partial \mathbf{\Psi}}{\partial \mathbf{s}_r}\right]^{p-1}$$

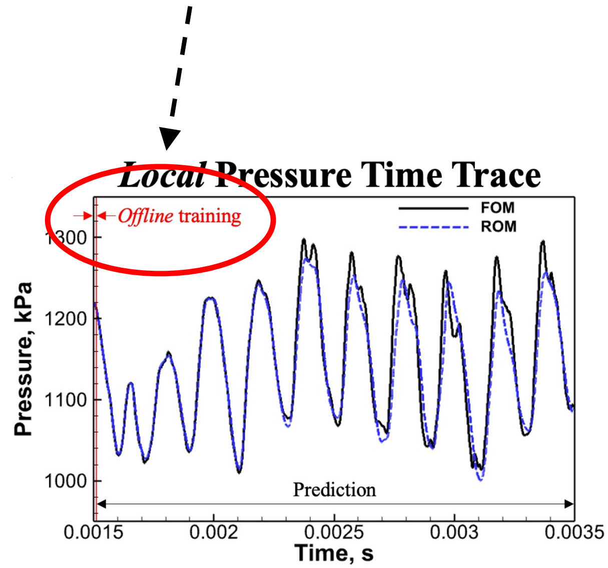
- Symmetrized at sub-iteration level!
- Notice two levels of scaling: **Q** and **S**

Reducing computational complexity of extreme multi-scale problems while preserving mathematical and physical fidelity



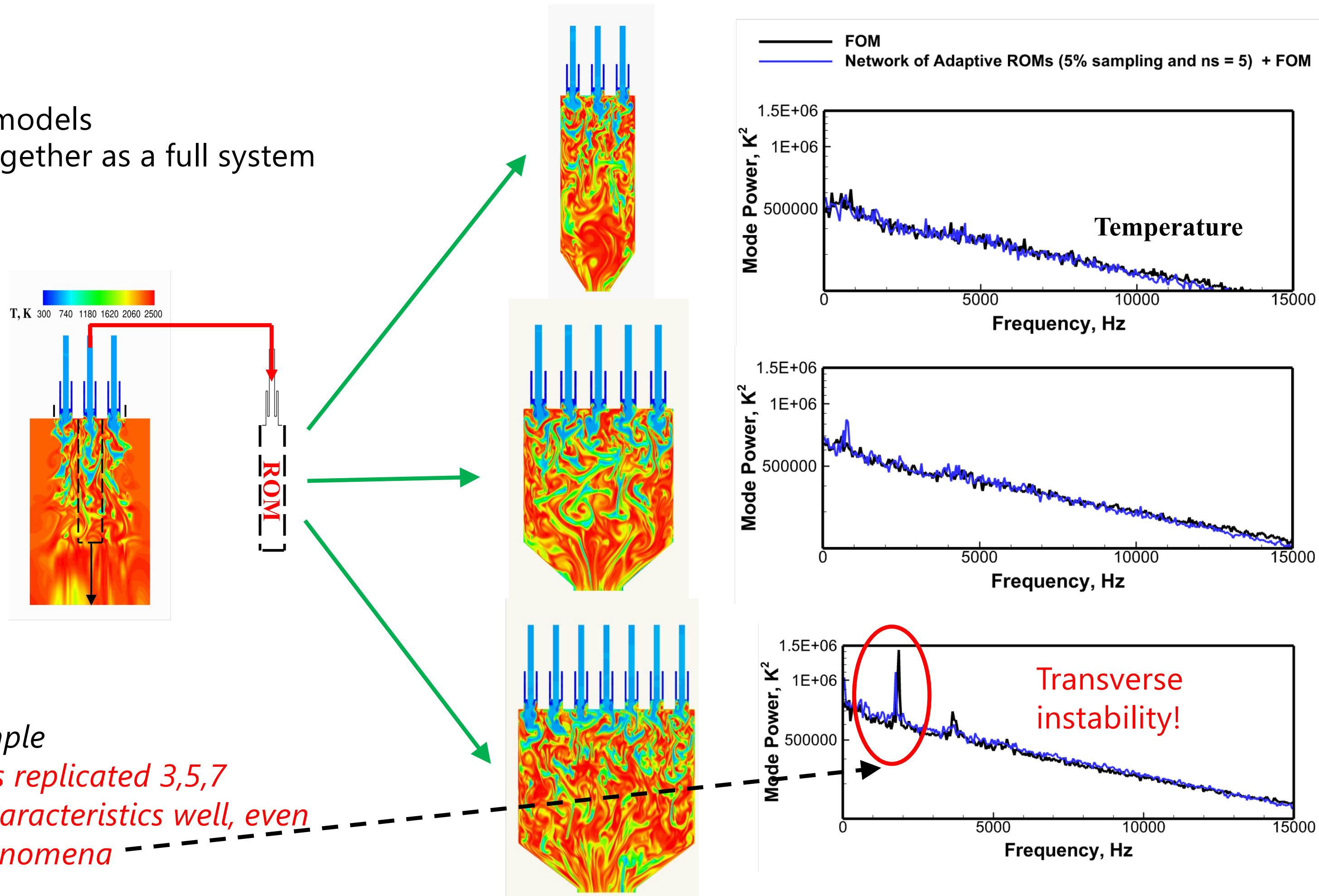
- In-situ adaptive sampling and projections to low-dimensional manifolds
- 2 orders of magnitude acceleration while preserving mathematical & physical fidelity
- Future state and parametric prediction of extremely complex chaotic flows with negligible off-line training





Enabling reduced complexity modeling without having access to full system simulations

- Train component based models
- Integrate components together as a full system
- Execute adaptive ROMs



Rocket combustor example

➔ component ROM was replicated 3,5,7 times. It predicts flow characteristics well, even capturing emergent phenomena