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Load redistribution near non-aligned fibre breaks in a two-dimensional unidirectional composite using break-influence superposition

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Statistical models for the failure process in unidirectional composite materials require knowledge of the distribution for strength of the fibres and details of the load redistribution in the vicinity of single and multiple fibre breaks. A common technique for examining these stress concentrations is the shear-lag approach, originally conceived by Cox [1], in which the matrix is assumed to transmit only shear stresses in between the fibres. Hedgepeth's classic solution [2] used this approach to construct a solution for stress concentrations in a two-dimensional unidirectional composite consisting of elastic fibres and matrix, with fibre breaks aligned transversely along the midline of an infinite sheet (for schematic and notation, see Fig. 1), for both static and dynamic stress concentration factors. Hedgepeth and Van Dyke [3] and Fichter [4] extended this model respectively to a three-dimensional unidirectional array (square and hexagonal) with aligned breaks, and to an aligned array of breaks intermittent with intact fibres.

These shear-lag solutions were limited in two respects: first, shear-lag theory precluded modelling of some of the mechanical aspects of the problem and, secondly, fibre breaks were restricted to a single line or plane, transverse to the fibre direction. Finite-element models have been developed to address these two shortcomings [5, 6], but the solutions, although able to capture many of the key mechanical aspects of the problem, have proven to be extremely computationally intensive. Generally these techniques are used to examine failure processes in small composites of fewer than 100 fibres, offering little real direct information on size scales of interest to designers. On the other hand, comparison of finite-element method results and shear-lag analyses has shown that the shear-lag solutions are quite adequate even in the cases of stiff matrices and high anisotropic fibres [5].

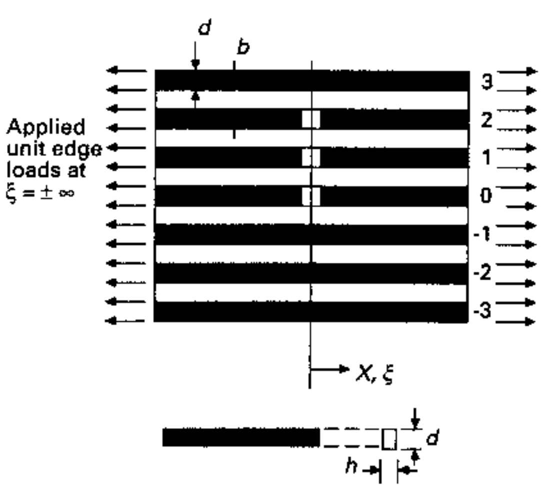

Figure 1 Schematic diagram of the problem solved by Hedgepeth [2], with notation.

Examination of statistical scalings occurring in failure processes of composite materials of size exceeding 10^6 fibre elements requires a more tractable solution approach for load redistribution. In isotropic materials several approaches have been developed to calculate stress profiles in the presence of arrays of small cracks, including that by Kachanov [7]. Kachanov's technique uses information about the stress distribution in a solid with only one crack to deduce the stress profile in the presence of many cracks, neglecting only the impact of the non-uniform traction of a crack on other cracks. The model presented here for fibre load profiles near breaks combines both Kachanov's model and Hedgepeth's model to simplify the analysis in large composite sheets. In the present setting the analysis is exact.

Referring to Fig. 1, the governing equation for the normalized displacement (U_n) of fibre n in Hedgepeth's model is

$$\frac{\partial^2 U_n}{\partial \xi^2} + U_{n+1} - 2U_n + U_{n-1} = 0 \qquad (1)$$

where the normalized constitutive law is

$$P_n = \partial U_n / \partial \xi \qquad (2)$$

and the normalized axial co-ordinate, displacement and load (with fibre cross-sectional area $A = dh$) are, respectively (see Fig. 1),

$$\xi = \frac{x}{(EAb/Gh)^{1/2}} \qquad (3)$$

$$U_n = \frac{U_n}{P(b/EAGh)^{1/2}} \qquad (4)$$

and

$$P_n = P_n/P \qquad (5)$$

In Hedgepeth's case of transversely aligned breaks the original normalized problem involved unit loads

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($P = 1$) applied on the fibres at the boundaries ($\xi = \pm\infty$) and with zero loads at the breaks. Load and displacement profiles $P_n(\xi)$ and $U_n(\xi)$ were found for the composite sheet. Two other intermediate solutions were required to yield this final solution: first, a "modified" problem where the applied loads were zero at the boundaries ($\xi = \pm\infty$) but compressive unity on the ends of the r broken fibres and, secondly, an "auxiliary" problem, which was the modified problem but with only one break (with normalized loads denoted L and displacements V). The auxiliary problem was first solved, and the reciprocal theorem was used to obtain the solution to the modified problem from the auxiliary solution. The full solution was obtained from that for the modified problem by simply adding unity to the load field and ξ to the displacement field. For the case of three breaks ($r = 3$) the full problem is shown in Fig. 1, the modified problem in Fig. 2a and the auxiliary problem in Fig. 2b.

The present solution approach takes a different direction from that of Hedgepeth, which is necessary because the greatly simplifying symmetry of the original problem no longer exists. Non-aligned breaks distort the displacements, so that displacements in the vicinity of a break are no longer symmetrical around the break. Consider the modified problem of three breaks as shown in Fig. 3a. Solutions of three distinct problems as in Fig. 3b (the dark spots are points of interest, not breaks) are used to solve the problem in Fig. 3a; this can in turn be used to solve the desired original problem of unit applied boundary loads with vanishing loads at the three break sites. The key to this approach, first developed by Kachanov [7] for cracks in isotropic materials, is that the influence of a break on the other break sites must be accounted for in the solution of the full problem in Fig. 3a. Determination of the stress concentrations due to each single break in succession at the other break sites first must be obtained.

Solutions to each problem in Fig. 3b are obtained by simple modifications (translation and scaling) of Hedgepeth's auxiliary solution for the static case, denoted $L_n(\xi)$ in his work. There a discretized Fourier transform was used to collapse the displacement equations to a single line, and application of the transformed boundary conditions yielded the solutions both in the transformed domain and for individual fibre displacements. The details can be seen in [2]. The resultant expression for load (found from Equation 2) is the solution to the problem of a unit negative load applied on either end of a single

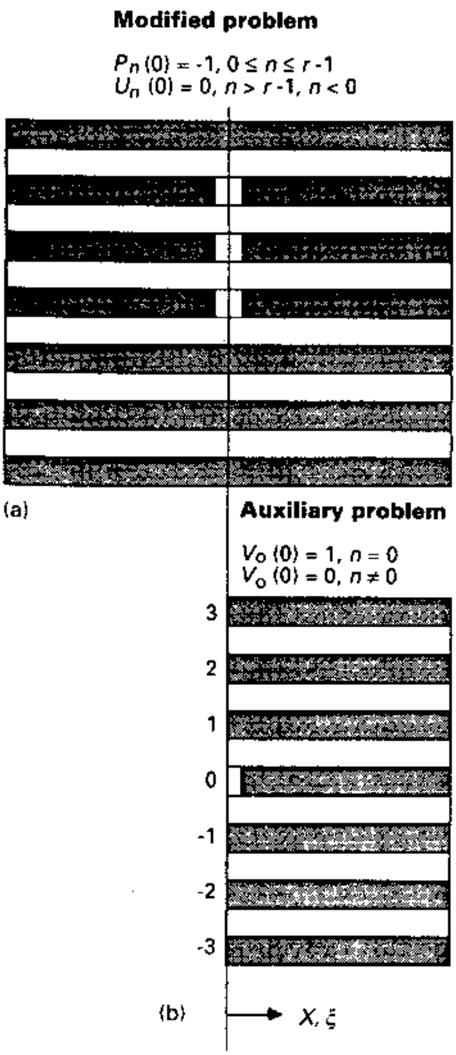

Figure 2 Schematic diagrams of (a) modified version of Hedgepeth's full problem in the case of three fibre breaks and (b) Hedgepeth's auxiliary problem [2].

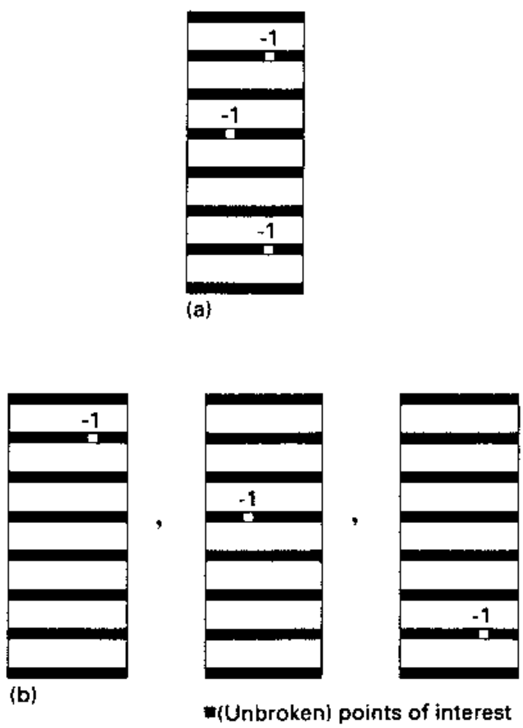

Figure 3 Solutions for the problems required in solution of stress concentrations with three arbitrarily located breaks, including (a) the required final solution and (b) the solutions to the individual breaks' concentrating effects (black markers indicate the locations of other breaks).

break located at $(n, \xi) = (0, 0)$. This load is given by

$$F_n(\xi) = -\tfrac{1}{2}\int_0^{\pi} \sin(\theta/2)\cos(n\theta)\exp[-2|\xi|\sin(\theta/2)]\,\mathrm{d}\theta \tag{6}$$

Because the composite sheet is assumed to be infinite, the solution for the stress concentrations generated by an arbitrarily located break can be obtained simply by shifting the above solution for loads to the appropriate break site. For a given break labelled i in fibre n_i, at position ξ_i the resultant equation for the load distribution is

$$F_{n-n_i}(\xi - \xi_i) = -\tfrac{1}{2}\int_0^{\pi} \sin(\theta/2)\cos[(n - n_i)\theta] \times \exp[-2|\xi - \xi_i|\sin(\theta/2)]\,\mathrm{d}\theta \tag{7}$$

We now define transmission factors (Λ_{ij}) as the load generated at break j as a result of a compressive load of negative unit intensity applied on the ends of break i (Fig. 4) in an infinite sheet with no other applied stresses. Kachanov [7, 8] defined these transmission factors as the average traction generated along the crack j line (where the crack will be) as a result of a unit-intensity traction applied on the faces of crack i. Since it is assumed here that normal stresses in a fibre do not vary radially, but only axially, and that a "crack" consists only of a pointwise break, the analysis is exact (it is approximate for the general cases discussed in [8]). The transmission factors for this anisotropic problem can be calculated directly from a simple modification of Equation 7. If the "address" (n, ξ) of breaks 1 and 2 are, respectively, (n_1, ξ_1) and (n_2, ξ_2), the transmission factor Λ_{12} is given by

$$\Lambda_{12} = -\tfrac{1}{2}\int_0^{\pi} \sin(\theta/2)\cos[(n_1 - n_2)\theta] \times \exp[-2|\xi_1 - \xi_2|\sin(\theta/2)]\,\mathrm{d}\theta \tag{8}$$

In general, transmission factors Λ_{ij} are given by

$$\Lambda_{ij} = -\tfrac{1}{2}\int_0^{\pi} \sin(\theta/2)\cos[(n_i - n_j)\theta] \times \exp[-2|\xi_i - \xi_j|\sin(\theta/2)]\,\mathrm{d}\theta \tag{9}$$

For a composite with r breaks, the transmission factors comprise an $r \times r$ symmetric matrix, i.e. $\Lambda_{ij} = \Lambda_{ji}$.

The full solution to the problem of many breaks can be expressed as a linear combination of solutions for the corresponding single-break problems. For the case of three breaks (shown in Fig. 3a) the corresponding problems are shown in Fig. 3b. The full solution to such a three-break problem with breaks at (n_1, ξ_1), (n_2, ξ_2) and (n_3, ξ_3) is

$$P_n(\xi) = K_1 F_{n-n_1}(\xi - \xi_1) + K_2 F_{n-n_2}(\xi - \xi_2) + K_3 F_{n-n_3}(\xi - \xi_3) \tag{10}$$

with unknown coefficients K_1, K_2 and K_3, determined as follows: the key step is to solve r simultaneous equations to enforce the boundary conditions at the break ends (such as Fig. 3a) in the modified problem; namely, the resultant compressive loads generated at the break sites of the composite in the presence of interacting cracks must equal -1 at each break end. For r breaks we substitute these boundary values into Equation 10 at each break site (n_i, ξ_i), and solve r simultaneous equations. For the case of three breaks

$$P_{n_i}(\xi_i) = K_1 F_{n_i-n_1}(\xi_i - \xi_1) + K_2 F_{n_i-n_2}(\xi_i - \xi_2) + K_3 F_{n_i-n_3}(\xi_i - \xi_3) = -1 \qquad i = 1, 2, 3 \tag{11}$$

which can be written more compactly in terms of transmission factors as

$$\begin{Bmatrix} -1 \\ -1 \\ -1 \end{Bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{12} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{13} & \Lambda_{23} & \Lambda_{33} \end{bmatrix} \begin{Bmatrix} K_1 \\ K_2 \\ K_3 \end{Bmatrix} \tag{12}$$

where the diagonal terms Λ_{11}, Λ_{22} and Λ_{33} are -1. For an arbitrary number r and arrangement of breaks, solutions for factors $K_1, \ldots, K_r$ will yield the full solution as in Equation 10 for a problem such as that shown in Fig. 3b.

An example of the effect of staggered breaks along the fibre direction is shown for the simplest case of two breaks on neighbouring fibres separated by axial distance δ in Fig. 5. For increasing distance δ between the two breaks, loads at locations (i), (ii) and (iii) (Fig. 5) are computed (by integrating the key equations numerically and following the solution procedure outlined above) and given in Table I. As this example shows, the resultant load profiles for non-aligned breaks are significantly different from those for aligned breaks. In the case of a relatively

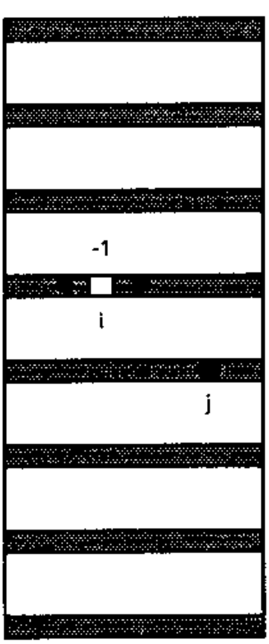

Figure 4 Two arbitrarily located breaks at i and j; solution for the effect of a unit negative load at the edges of the break at i on the point j must be found.

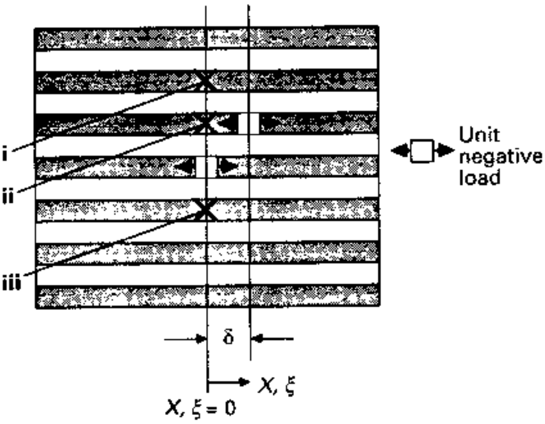

Figure 5 Notation for the two-break example, with breaks separated axially by distance δ, and with points of interest (i), (ii) and (iii).

TABLE I Loads at locations (i), (ii) and (iii) generated in a composite with two staggered breaks separated by distance δ, as shown in Fig. 5

δ	Load at (i)	Load at (ii)	Load at (iii)
0.000	0.600	−1.000	0.600
0.100	0.446	−0.708	0.541
0.200	0.341	−0.503	0.498
0.500	0.170	−0.148	0.421
1.000	0.072	0.100	0.362
2.000	0.041	0.254	0.326
3.000	0.047	0.297	0.321
4.000	0.053	0.313	0.322
5.000	0.057	0.320	0.324
10.000	0.064	0.330	0.330
100.000	0.067	0.333	0.333

compliant matrix ($E/G = 100$) and a 50% fibre volume fraction ($b = d$), the normalized break spacing $\delta = 1$ corresponds to an actual length of 10 fibre diameters (normalization of δ is as the normalization of axial co-ordinate ξ given by Equation 3). Thus, Table I gives values for resultant loads at locations (i), (ii) and (iii) in such a material for a misalignment δ ranging from 1 to 1000 fibre diameters. The results for $\delta = 0$ correspond to a Hedgepeth-type solution and match his results for the two-break case [2].

Two key points can be observed from this simple example, for the cases of closely spaced and widely spaced breaks, respectively. For non-aligned, closely spaced breaks, it can be seen that even for a slight staggering ($\delta = 0.1$, corresponding to only one fibre diameter in the material described above), the resultant loads at points (i) and (iii) are significantly lessened. This effect is important, as statistically it would not be expected to find perfectly aligned breaks in a composite. The effect of axial dispersion of closely-spaced breaks on resultant load concentrations is clearly important. In the case of widely spaced breaks, however, the interaction between breaks is small, and interestingly can also be negative. That is, for a range of break spacings, the effect of surrounding breaks on a given break can be to lessen the severity of the resultant load concentrations, or to "shield" the break. This can be seen in Table I for the two-break example. The loads at (i) are actually less than those resulting from a single break at $(n, \xi) = (0, 0)$ for two breaks separated by δ, where $\delta > 2$. As the distance between breaks becomes even larger, as shown in Table I for the case of $\delta = 1000$, the resultant loads at (i), (ii) and (iii) match Hedgepeth's result [2] for a single break at $(n, \xi) = (0, 0)$.

This method of analysis greatly reduces the computation in the simplest case of aligned breaks [2–4], and provides a fast method of solution for the overloads in a unidirectional composite with an arbitrary array of breaks. The method requires three basic steps: first, computation of the transmission factors Λ_{ij} through (numerical) integrations given by Equation 9 (which can be computed initially and stored for a sufficiently fine grid of potential values of $n_i - n_j$ and $\xi_i - \xi_j$); secondly, inversion of the $r \times r$ interaction matrix in Equation 12 to solve for factors $K_1, \ldots, K_r$; and, thirdly, construction of the full solution for the loads in the sheets through the shifted solutions of the single-break problems, as in Equation 10. The solution is exact for the stated problem, satisfying both the equilibrium equations at every point and all of the boundary conditions. Numerical efficiency results from the fact that the break–break interactions become negligible beyond a certain distance.

In future work this solution will be extended to other geometries. The ultimate goal of the research is to perform Monte-Carlo simulations for strength using the mechanical model given here, for comparison with various statistical models (for example, those in [9, 10]). A key aspect to examine is variability in fibre strength and its effect on the dispersion and shielding phenomena in the strength distribution of the composite.

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